When does the box dimension of a fractal exist?

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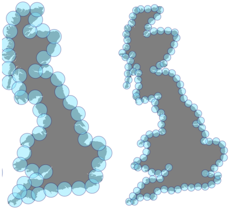


[1]Based on joint work with Alex Rutar, https://arxiv.org/abs/2406.12821 Picture by Prokofiev, CC BY-SA 3.0

Box dimension

- The least number of balls of radius r needed to cover the British coastline E scales like $N_r(E) \approx r^{-1.2}$ for a range of r. The 'fractal dimension' of the British coastline is ≈ 1.2.
- The lower and upper box (Minkowski) dimensions of a non-empty bounded set $E \subset \mathbb{R}^d$ are

$$\underline{\dim}_{\mathrm{B}} E = \liminf_{r \to 0} \frac{\log N_r(E)}{\log(1/r)}, \qquad \overline{\dim}_{\mathrm{B}} E = \limsup_{r \to 0} \frac{\log N_r(E)}{\log(1/r)}.$$

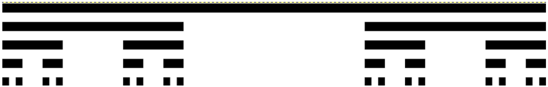
- Always $\underline{\dim}_{\mathrm{B}} E \leq \overline{\dim}_{\mathrm{B}} E$. If $\underline{\dim}_{\mathrm{B}} E = \overline{\dim}_{\mathrm{B}} E$, the common value $\dim_{\mathrm{B}} E$ is the box dimension of E. In this case the extent to which E 'fills up space' does not vary much at different scales.
- **Question:** for which classes of sets does the box dimension exist?

Dynamically invariant sets

Theorem (Barreira 1996 / Gatzouras–Peres 1997)

If $f \colon M \to M$ is an expanding **conformal** C^1 map of a Riemannian manifold and a **compact** $\Lambda \subseteq M$ satisfies $f(\Lambda) = \Lambda$ and $f^{-1}(\Lambda) \cap U \subseteq \Lambda$ for a neighbourhood U of Λ, then $\dim_{\mathrm{B}} \Lambda$ exists and coincides with Hausdorff dimension.

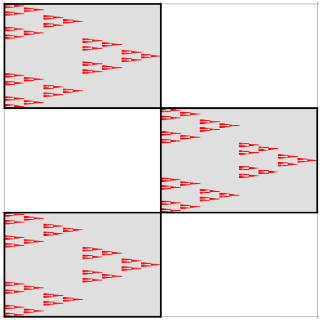
Self-similar and self-conformal sets Λ satisfy $\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} \Lambda$.



The middle-third Cantor set is self-similar and invariant for $f(x) = 3x \mod 1$. Picture based on one by Thefrettinghand, CC BY-SA 3.0

Non-conformal dynamics

Bedford (1984) and McMullen (1984) constructed compact sets invariant under non-conformal toral endomorphisms such as $(x, y) \mapsto (2x \mod 1, 3y \mod 1)$, with distinct Hausdorff and box dimension.



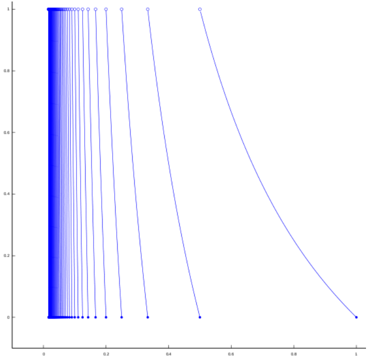
Non-conformal dynamics

- Jurga (2023) constructed a compact set Λ invariant for $(x, y) \mapsto (2x \mod 1, 12y \mod 1)$ with $\underline{\dim}_B \Lambda < \overline{\dim}_B \Lambda$.
- The example is a sub-self-affine set ($\Lambda \subset \bigcup_i S_i(\Lambda)$ for finitely many affine contractions S_i), whereas Bedford–McMullen carpets are self-affine sets ($\Lambda = \bigcup_i S_i(\Lambda)$).
- **Folklore conjecture:** the box dimension of every self-affine set should exist.

The Gauss map

The Gauss map $\mathcal{G}\colon [0,1) \to [0,1)$ is defined by

$$\mathcal{G}(x) = \begin{cases} x^{-1} - \lfloor x^{-1} \rfloor & : 0 < x < 1 \\ 0 & : x = 0. \end{cases}$$



Picture by Adam majewski,

The Gauss map

Typical invariant sets are numbers whose continued fraction expansions are restricted to some $I \subset \mathbb{N}$:

$$\Lambda_I := \left\{ z \in (0,1) \setminus \mathbb{Q} : z = \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{\ddots}}}, b_n \in I \text{ for all } n \in \mathbb{N} \right\}$$

satisfies $\mathcal{G}(\Lambda_I) = \Lambda_I$. If I is infinite then F_I is non-compact.

Theorem

- Mauldin & Urbański ('96, '99): there exists $I \subset \mathbb{N}$ with $\dim_{\mathrm{H}} \Lambda_I < \dim_{\mathrm{B}} \Lambda_I$.
- B.–Rutar ('24+): there exists $I \subset \mathbb{N}$ with $\dim_{\mathrm{H}} \Lambda_I < \underline{\dim}_{\mathrm{B}} \Lambda_I < \overline{\dim}_{\mathrm{B}} \Lambda_I$. In particular, the box dimension of Λ_I does not exist.

Infinite conformal IFS (Mauldin & Urbański, '96)

A conformal iterated function system is a countable family of uniformly contracting, conformal maps $\{S_i \colon X \to X\}_{i \in I}$ on a 'nice' (e.g. non-empty convex compact) set $X \subset \mathbb{R}^d$. For continued fraction sets the maps are $\{x \mapsto (b+x)^{-1} : b \in I\}$. We always assume:

- **Open set condition:** $\mathrm{Int}(X) \neq \varnothing$ and $\bigcup_{i \in I} S_i(\mathrm{Int}(X)) \subseteq \mathrm{Int}(X)$ with the union disjoint.
- **Bounded distortion**

The **limit set** is the largest set $\Lambda \subseteq X$ satisfying

$$\Lambda = \bigcup_{i \in I} S_i(\Lambda)$$

(it is generally non-compact).

Hausdorff and box dimensions

For $w \in I^k$ let R_w be the smallest possible Lipschitz constant for $S_w := S_{w_1} \circ \cdots \circ S_{w_k}$ and define the **pressure function**

$$P(t) := \lim_{k \to \infty} \frac{1}{k} \log \sum_{w \in I^k} R_w^t,$$

Theorem (Mauldin–Urbański, '96, '99)

- $\dim_{\mathrm{H}} \Lambda = \inf\{t > 0 : P(t) < 0\}$
- $\overline{\dim}_{\mathrm{B}} \Lambda = \max\{\dim_{\mathrm{H}} \Lambda, \overline{\dim}_{\mathrm{B}} F\}$, where F is the set of **fixed points** of the contractions (or for the continued fraction sets Λ_I we can take $F = \{1/b : b \in I\}$).

Bounds for lower box dimension

Bounds for $\underline{\dim}_{\mathrm{B}} \Lambda$ that are immediate from Mauldin–Urbański:

$$\max\{\dim_{\mathrm{H}} \Lambda, \underline{\dim}_{\mathrm{B}} F\} \leq \underline{\dim}_{\mathrm{B}} \Lambda \leq \max\{\dim_{\mathrm{H}} \Lambda, \overline{\dim}_{\mathrm{B}} F\} = \overline{\dim}_{\mathrm{B}} \Lambda.$$

Theorem (B.–Rutar, '24+)

The box dimension of Λ exists if and only if these bounds coincide.

In fact $\underline{\dim}_{\mathrm{B}} \Lambda$ is **not** a function of $\dim_{\mathrm{H}} \Lambda$, $\underline{\dim}_{\mathrm{B}} F$, $\overline{\dim}_{\mathrm{B}} F$:

Theorem (B.–Rutar, '24+)

The trivial lower bound for $\underline{\dim}_{\mathrm{B}} \Lambda$ is sharp, and a (non-trivial) sharp upper bound is

$$\underline{\dim}_{\mathrm{B}} \Lambda \leq \dim_{\mathrm{H}} \Lambda + \frac{(\overline{\dim}_{\mathrm{B}} F - \dim_{\mathrm{H}} \Lambda)(d - \dim_{\mathrm{H}} \Lambda)\,\underline{\dim}_{\mathrm{B}} F}{d\,\overline{\dim}_{\mathrm{B}} F - \dim_{\mathrm{H}} \Lambda\,\underline{\dim}_{\mathrm{B}} F}.$$

An asymptotic formula

These results follow from a formula for $\underline{\dim}_{\mathrm{B}} \Lambda$ in terms of the whole function

$$r \mapsto s_F(r) := \frac{\log N_r(F)}{\log(1/r)}.$$

(and on the contraction ratios, but only via $\dim_{\mathrm{H}} \Lambda$).
Define the weighted average

$$\Psi(r,\theta) := (1-\theta)\dim_{\mathrm{H}} \Lambda + \theta s_F(r^\theta)$$

and $\psi(r) := \sup_{\theta \in (0,1]} \Psi(r,\theta)$.

Theorem (B.–Rutar, '24+)

$$\frac{\log N_r(\Lambda)}{\log(1/r)} - \psi(r) \xrightarrow[r\to 0]{} 0, \quad \text{hence} \quad \underline{\dim}_{\mathrm{B}} \Lambda = \liminf_{r\to 0} \psi(r).$$

Thank you for listening!