Fractal measures with slow Fourier decay

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Fourier transform of measures

- **Fourier transform** of a compactly supported Borel probability measure μ on $\mathbb{R}$ is

$$\widehat{\mu}\colon \mathbb{R} \to \mathbb{C}, \qquad \widehat{\mu}(\xi) := \int_{\mathbb{R}} e^{-2\pi i \xi x} d\mu(x).$$

- μ is **Rajchman** if $|\widehat{\mu}(\xi)| \to 0$ as $|\xi| \to \infty$.
  For Rajchman measures one can ask about the **rate** of decay.

Self-similar and self-conformal measures

- Let $\Phi = \{S_i \colon [0,1] \to [0,1]\}_{i \in I}$ be a finite C^2 IFS with $0 < c \leq |S_i'(x)| \leq C < 1$ for all i and $x \in [0,1]$.
- Given weights $p_i > 0$, $\sum_i p_i = 1$, the **self-conformal measure** μ satisfies for all Borel $A \subseteq \mathbb{R}$,

$$\mu(A) = \sum_{i \in I} p_i \mu(S_i^{-1}(A)).$$

  Assume not all maps share a common fixed point, so μ is non-atomic, supported on an uncountable set (the attractor).
- If each contraction is affine then μ is **self-similar**.

Fourier decay of self-similar measures

- Cantor–Lebesgue measure (equal weights, IFS $\{T_1(x) = x/3, T_2(x) = (x+2)/3\}$) is not Rajchman:
$$0 \neq \widehat{\mu}(1) = \widehat{\mu}(3) = \widehat{\mu}(9) = \cdots.$$
- "Almost every" self-similar measure satisfies $|\widehat{\mu}(\xi)| \lesssim |\xi|^{-\eta}$ for some $\eta > 0$.
- Paukkonen–Sahlsten–Streck (in progress) construct inhomogeneous self-similar measures with
$$\limsup_{\xi \to \infty} \frac{|\widehat{\mu}(\xi)|}{(\log \xi)^{-\eta}} \approx 1.$$

Question

Does every Rajchman self-similar measure in $\mathbb{R}$ satisfy $|\widehat{\mu}(\xi)| \leq (\log|\xi|)^{-\eta}$ for some $\eta > 0$ for all large enough $|\xi|$?

Self-similar measures with slow Fourier decay

Theorem (Baker–B., '26+)

Answer is **no**. For every $\phi\colon [0,\infty) \to (0,1]$ with $\phi(\xi) \to 0$ as $\xi \to \infty$, there is a Rajchman self-similar measure μ (depending on ϕ) satisfying

$$\limsup_{\xi \to \infty} \frac{|\widehat{\mu}(\xi)|}{\phi(\xi)} > 0.$$

- Cannot replace lim sup by lim inf. Self-similar measures are Frostman, so there are $\eta > 0$ and $\xi_1 < \xi_2 < \cdots \to \infty$ with $|\widehat{\mu}(\xi_n)| \leq \xi_n^{-\eta}$ for all n.
- Use equal weights with the IFS

$$\left\{ S_0(x) = \frac{x}{10},\ S_1(x) = \frac{x+1}{10},\ S_2(x) = \frac{x+t}{10} \right\}$$

where t is Liouville, very well approximated by rationals $p/10^n$.

Real-analytic IFSs

- A C^2 IFS $\{S_i\}$ is called **linear** if $S_i''(x) = 0$ for all x in the **attractor**.
- For analytic IFSs, Algom–Rodriguez Hertz–Wang ('23+) observed a dichotomy: either
  1. there is a self-similar IFS $\{T_i\}$ and analytic diffeomorphism F with $\{T_i\} = \{F \circ S_i \circ F^{-1}\}$, or
  2. the IFS is not conjugate to any linear IFS by any C^2 diffeomorphism.
- Case (1): every self-conformal measure μ can be written $F\nu$ where ν is self-similar. If some S_i is non-affine then $|\widehat{\mu}(\xi)| \lesssim |\xi|^{-\eta}$ by Algom–Chang–Wu–Wu (25') / Baker–B. ('25).
- Case (2): $|\widehat{\mu}(\xi)| \lesssim |\xi|^{-\eta}$ by AHW ('23+) / Baker–Sahlsten ('23+).

Analytic IFSs

Piecing things together:

Analytic Theorem

If $\{S_i\colon [0,1] \to [0,1]\}$ is an IFS of analytic maps (no separation assumptions) and there is i such that S_i is not affine, then every self-conformal measure satisfies $|\widehat{\mu}(\xi)| \lesssim |\xi|^{-\eta}$.

Question (Sahlsten, Proceedings of FARF4, paraphrased)

Does some analogue hold for IFSs with lower regularity?

Lower regularity

Theorem (Algom–Ben Ovadia–Rodriguez Hertz–Shannon ('26 and in progress))

The dichotomy breaks down in lower regularity. There are C^∞ IFSs which are linear but not conjugate to self-similar, and resulting self-conformal measures are sometimes Rajchman but not always.

Henceforth we work with C^2-conjugate-to-self-similar IFSs.

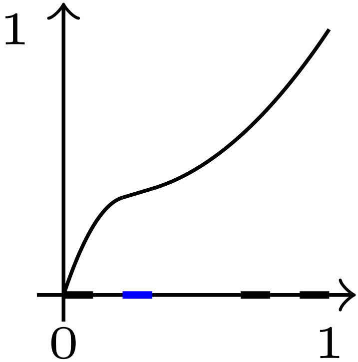
Theorem (Baker–B. ('26+))

Let μ be a self-conformal measure for a C^2 IFS $\{S_i \colon [0,1] \to [0,1]\}$, C^2-conjugate-to-self-similar. Assume there is i with $S_i''(x) \neq 0$ for **all** x. Then $|\widehat{\mu}(\xi)| \lesssim |\xi|^{-\eta}$.

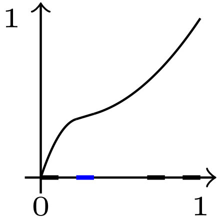
Proof idea: second derivative of the conjugacy vanishes nowhere, so apply BB ('25) / ACWW ('25).

Non-Rajchman example (Baker–B. ('26+))

Let ν be Cantor–Lebesgue on $[0,1]$ (IFS $\{T_1, T_2\}$) and $\mu = F\nu$. Here $F\colon [0,1] \to [0,1]$ is an increasing C^∞ diffeomorphism, affine on $[2/9, 1/3]$, with $F''(x) \neq 0$ for $x \in [0, 2/9) \cup (3/9, 1]$.



Non-Rajchman example



- Can choose F so μ is self-conformal for the C^∞ IFS $\{S_i := F \circ T_i \circ F^{-1}\}$ (equal weights), and $S_1''(x) \neq 0$, $S_2''(x) \neq 0$ apart from finitely many x.
- μ contains an affine copy of ν and is **non-Rajchman**.
- But $(S_1 \circ S_2)''(x) = 0$ for all x.

Refined Question

So we want a nonlinearity assumption on **iterates** of the IFS.

Refined Question

Let $\{S_i\}$ be a C^2 IFS. Assume that for all $i_1, \ldots, i_n$ we have $(S_{i_1} \circ \cdots \circ S_{i_n})''(x) \neq 0$ except at finitely many x. Does it follow that every self-conformal μ satisfies $|\widehat{\mu}(\xi)| \lesssim |\xi|^{-\eta}$?

Theorem (Baker–B. ('26+))

If $\{S_i\}$ is a C^2 and C^2-conjugate-to-self-similar IFS such that for all $i_1, \ldots, i_n$, $(S_{i_1} \circ \cdots \circ S_{i_n})''(x) \neq 0$ except at finitely many x, then every self-conformal measure is **Rajchman**.

Proof idea: the conjugate self-similar measure of the zero set of the second derivative of the conjugacy is 0. Apply BB ('25) / ACWW ('25) away from this set.

Answering Refined Question

Answer to Refined Question is **no**.

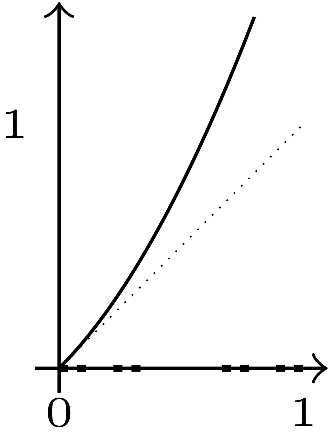
Theorem (Baker–B. ('26+))

For every $\phi\colon [0,\infty) \to (0,1]$ such that $\phi(\xi) \to 0$ as $\xi \to \infty$, there is a C^∞ IFS $\{S_i\}$ such that

- $\{S_i\}$ is C^∞-conjugate-to-self-similar,
- for all $i_1, \ldots, i_n$, $(S_{i_1} \circ \cdots \circ S_{i_n})''(x) \neq 0$ except at finitely many x, and
- $\{S_i\}$ generates some (Rajchman) self-conformal measure μ satisfying

$$\limsup_{\xi\to\infty} \frac{|\widehat{\mu}(\xi)|}{\phi(\xi)} > 0.$$

Example with slow Fourier decay



Example with slow Fourier decay

- Let ν be Cantor–Lebesgue (IFS $\{T_1, T_2\}$).
- Let
$$F\colon [0,1] \to \mathbb{R}, \quad F(x) := \begin{cases} x + \exp(-\exp(x^{-2})), & x \in (0,1], \\ 0, & x = 0. \end{cases}$$
- $F\nu$ is self-conformal for the C^∞ IFS $\{S_1, S_2\} := \{F \circ T_1 \circ F^{-1}, F \circ T_2 \circ F^{-1}\}$.
- For all $i_1, \ldots, i_n \in \{1,2\}^n$, $(S_{i_1} \circ \cdots \circ S_{i_n})''(x) \neq 0$ except at finitely many x.
- $$\limsup_{\xi \to \infty} \frac{|\widehat{F\nu}(\xi)|}{(\log \log \xi)^{-1}} > 0.$$

Proof idea

- Let integers $k_1 \leq k_2 \leq \cdots$ be s.t. $3^{k_n} \exp(-\exp(3^{2n})) \approx 0.01$.
- $\widehat{F\nu}(3^{k_n}) = \int_{\mathbb{R}} e^{-2\pi i 3^{k_n} F(x)} d\nu(x) = \int_0^{3^{-n}} \cdots + \int_{2(3^{-n})}^1 \cdots$.
- $\left| \int_0^{3^{-n}} \cdots \right| \approx \nu([0, 3^{-n}]) = 2^{-n}$.
- By methods from Baker–B. ('25), if $\mathcal{F}$ is a family of C^2 maps $f \colon [0,1] \to \mathbb{R}$ with $f''(x) \neq 0 \, \forall f \in \mathcal{F} \, \forall x \in [0,1]$, and $\sup_{f \in \mathcal{F}} \max(||f'||_\infty, ||f''||_\infty) < \infty$, then there is $\eta > 0$ such that $\forall f \in \mathcal{F}$,
$$|\widehat{f\nu}(\xi)| \lesssim \big(\min_{0 \leq x \leq 1} |f''(x)|\big)^{-1} |\xi|^{-\eta}.$$
- $\left| \int_{2(3^{-n})}^1 \cdots \right| \leq 2^{-n} \sum_{\underline{i} \in \{1,2\}^n \setminus \{1^n\}} |\widehat{(F \circ T_{\underline{i}})\nu}(3^{k_n})| \ll 2^{-n}$.
- So $|\widehat{F\nu}(3^{k_n})| \approx 2^{-n} \gg (\log\log(3^{k_n}))^{-1}$.

Proof idea for $\limsup_{\xi\to\infty}|\widehat{\mu}(\xi)|/\phi(\xi) > 0$.

- Let
$$W(x) := \begin{cases} e^{-x^{-2}}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$
- Let $c_n \searrow 0$ fast (depending on ϕ), $r_n \searrow 0$ slowly. Define F by $F(0) = 0$, $F'(0) = 1$, and
$$F''(x) := \sum_{n=1}^{\infty} c_n W(x - r_{n+1}).$$
- Then F is C^∞ and $0 < f''(r_{n+1}) \ll f''(r_n)$. Set $\mu := F\nu$ and break up the integral into $[0, r_{n+1}]$, $[r_{n+1}, r_n]$, $[r_n, 1]$.
- To show $(S_{i_1} \circ \cdots \circ S_{i_n})''(x) \neq 0$ except at finitely many x, have to show that a **typical** choice of the c_n works.

Thank you for listening!