Fourier decay for nonlinear self-conformal measures

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[1] Joint work with Simon Baker (Loughborough University):
(1) "Polynomial Fourier decay for fractal measures and their pushforwards," *Math. Ann.* **392** (2025), 209–261.
(2) "Self-similar and self-conformal measures with slow Fourier decay," preprint (2026), https://arxiv.org/abs/2602.05593

Fourier transform of measures

- **Fourier transform** of a compactly supported Borel probability measure μ on $\mathbb{R}$ is the

$$\widehat{\mu}\colon \mathbb{R} \to \mathbb{C}, \qquad \widehat{\mu}(\xi) := \int_{\mathbb{R}} e^{-2\pi i \xi x} d\mu(x).$$

- Questions:
  - Is μ **Rajchman** ($|\widehat{\mu}(\xi)| \to 0$ as $|\xi| \to \infty$)?
  - Does μ have **polynomial Fourier decay**

$$|\widehat{\mu}(\xi)| \lesssim |\xi|^{-\sigma}?$$

- Riemann–Lebesgue lemma: absolutely continuous measures are Rajchman.

Self-similar and self-conformal measures

- Let $\Phi = \{S_i \colon [0,1] \to [0,1]\}_{i \in I}$ be an iterated function system (IFS) of smooth injective contractions. Assume not all maps share a common fixed point.
- Given weights $p_i > 0$, $\sum_i p_i = 1$, the **self-conformal measure** μ is non-atomic and satisfies

$$\mu(A) = \sum_{i \in I} p_i \mu(S_i^{-1}(A)).$$

  Its support (the **IFS attractor**) is uncountable.
- If each contraction is **affine** then μ is called **self-similar**.
- The Cantor–Lebesgue measure ($(1/2, 1/2)$ measure for IFS $\{T_1(x) = x/3, T_2(x) = (x+2)/3\}$) is not Rajchman:

$$0 \neq \widehat{\mu}(1) = \widehat{\mu}(3) = \widehat{\mu}(9) = \cdots.$$

- Erdős ('40), Kahane ('71), Solomyak ('22): "almost every" self-similar measure has polynomial Fourier decay.

Nonlinear images of self-similar measures

- Does **nonlinearity** in conformal IFS $\to$ polynomial Fourier decay?
- Kaufman ('84): the image of Cantor–Lebesgue by $x \mapsto x^2$ has polynomial Fourier decay.
- Mosquera–Shmerkin ('18) extended to nonlinear pushforwards of **homogeneous** self-similar measures (all contraction ratios equal).

Theorem (Baker–B. ('25) / Algom–Chang–Wu–Wu ('25))

Let μ be a self-similar measure on $[0,1]$. Assume $F\colon [0,1] \to \mathbb{R}$ is *either* (i) **real-analytic and non-affine** *or* (ii) C^2 **with** $F''(x) > 0$ $\forall x \in [0,1]$. Then $F\mu$ has polynomial Fourier decay.

No homogeneity or separation assumptions needed.

Linear IFSs

- Two IFSs $\{S_i\}$ and $\{T_i\}$ are called **conjugate** if there is a diffeomorphism F with $\{T_i\} = \{F \circ S_i \circ F^{-1}\}$.
- A C^2 IFS $\{S_i\}$ is called **linear** if $S_i''(x) = 0$ for all x in the **attractor**.
- Algom – Ben Ovadia – Rodriguez Hertz – Shannon ('26) proved the existence of linear IFSs which are not conjugate to self-similar.
- Fourier decay for self-conformal measures for such IFSs (and their nonlinear images) would be an interesting topic for **future research**.

Non-conjugate-to-linear IFSs

- Example: inverse branches of the Gauss map, $\{(x+b)^{-1} : b \in I\}$ for finite $I \subset \{2, 3, 4, \dots\}$. Self-conformal measures are supported on numbers with contined fraction expansions in I.
- Polynomial Fourier decay holds under good separation conditions Kaufman ('80), Queffélec–Ramaré ('03), Jordan–Sahlsten ('16), Bourgain–Dyatlov ('17), Sahlsten–Stevens ('24).

Theorem (Baker – Sahlsten ('23+) / Algom – Rodriguez Hertz – Wang ('23+))

If a C^2 IFS $\{S_i : [0,1] \to [0,1]\}$ satisfies a **uniform non-integrability (UNI)** assumption then all self-conformal measures have polynomial Fourier decay (no separation assumptions needed).

Key step: proving a spectral gap type result for a family of transfer operators using a disintegration method.

Analytic IFSs

- For **real analytic** IFSs, AHW ('23+) observed a dichotomy: either
  1. the IFS is not conjugate to any linear IFS by any C^2 diffeomorphism, or
  2. the IFS is conjugate to a self-similar IFS by an analytic diffeomorphism F.
- In case (1), AHW ('22) observed that UNI holds, so self-conformal measures have polynomial Fourier decay by BS ('23+) / AHW ('23+).
- In case (2), trivial to observe that every self-conformal measure μ can be written $F\nu$ where ν is self-similar.
  If the analytic IFS contains a non-affine map then F is non-affine, so by BB ('25) / ACWW ('25), μ has polynomial Fourier decay.

Analytic IFSs

Piecing everything together:

Analytic Theorem

If $\{S_i\colon [0,1] \to [0,1]\}$ is an IFS of analytic maps and there is i such that S_i is not affine, then every self-conformal measure has polynomial Fourier decay.

No separation assumptions needed.

Application: normal numbers and equidistribution

Let μ be as in Analytic Theorem. Then

- Using Davenport–Erdős–LeVeque ('63): μ-almost every x is *normal* in every base $b \geq 2$: the base-b expansion observes all finite words on $\{0, 1, \ldots, b-1\}$ with the expected frequency.
- Using Pollington–Velani–Zafeiropoulos–Zorin ('22): for an interval $I \subseteq [0, 1)$, for μ-a.e. x, as $N \to \infty$,

$$\frac{1}{N} \cdot \#\{1 \leq n \leq N : b^n x \mod 1 \in I\} = \text{length}(I) + \frac{E(N)}{N^{1/2}}$$

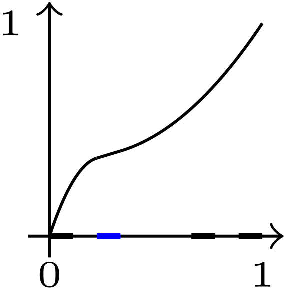
  for some $E(N) \lesssim (\log(N+1))^3$.
  Cannot hope to improve the exponent $1/2$, because **random** sequences see this exponent.

Lower regularity

Question (Sahlsten ('23+))

If $\{S_i\}$ is a $C^{1+\alpha}$ IFS and some S_i is non-affine, must every self-conformal measure have polynomial Fourier decay?

No.



Non-Rajchman example (Baker–B. ('26+))

- Let ν be Cantor–Lebesgue on $[0,1]$ (for the IFS $\{T_1, T_2\}$) and $\mu = F\nu$. Here $F\colon [0,1] \to [0,1]$ is an increasing C^∞ diffeomorphism, affine on $[2/9, 1/3]$, with $F''(x) \neq 0$ for $x \in [0, 2/9) \cup (3/9, 1]$.
- Can choose F so μ is a self-conformal measure for the C^∞ IFS $\{S_i := F \circ T_i \circ F^{-1}\}$ (weights $(1/2, 1/2)$), and $S_1''(x) \neq 0$, $S_2''(x) \neq 0$ apart from finitely many x.
- μ contains an affine copy of ν and is **non-Rajchman**.
- But $(S_1 \circ S_2)''(x) = 0$ for all x.
  This suggests we want a nonlinearity assumption on **iterates** of the IFS.

Nonlinearity assumption at all points

Theorem (Baker–B. ('26+))

Let μ be a self-conformal measure for a C^2 IFS $\{S_i \colon [0,1] \to [0,1]\}$ which is C^2-conjugate-to-self-similar. Assume there exist $i_1, \ldots, i_n$ such that $(S_{i_1} \circ \cdots \circ S_{i_n})''(x) \neq 0$ for **all** x. Then μ has polynomial Fourier decay.

Proof idea: the nonlinearity assumption for **all** x implies the second derivative of the conjugacy vanishes nowhere.
So we can apply BB ('25) / ACWW ('25).

Refined Question

Refined Question (Baker–B. ('26+))

Let $\{S_i\}$ be a $C^{1+\alpha}$ IFS and assume that for all $i_1, \ldots, i_n$ we have $(S_{i_1} \circ \cdots \circ S_{i_n})''(x) \neq 0$ except at finitely many x. Does it follow that all self-conformal measures have polynomial Fourier decay?

Theorem (Baker–B. ('26+))

If $\{S_i\}$ is a C^2 and C^2-conjugate-to-self-similar IFS such that for all $i_1, \ldots, i_n$, $(S_{i_1} \circ \cdots \circ S_{i_n})''(x) \neq 0$ except at finitely many x, then all self-conformal measures are **Rajchman**.

Proof idea: the conjugate self-similar measure of the zero set of the conjugacy is 0.
Apply BB ('25) / ACWW ('25) away from this set.

Slow Fourier decay

Answer to Refined Question is **no**.

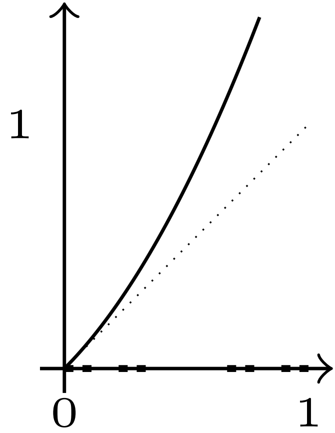
Theorem (Baker–B. ('26+))

For every $\phi\colon [0,\infty) \to (0,1]$ such that $\phi(\xi) \to 0$ as $\xi \to \infty$, there is a C^∞ IFS $\{S_i\}$ such that for all $i_1, \ldots, i_n$, $(S_{i_1} \circ \cdots \circ S_{i_n})''(x) \neq 0$ except at finitely many x, generating a self-conformal measure μ satisfying

$$\limsup_{\xi \to \infty} \frac{|\widehat{\mu}(\xi)|}{\phi(\xi)} > 0.$$

- The $\{S_i\}$ we construct is C^∞-conjugate-to-self-similar, so by previous theorem, μ is Rajchman.
- Cannot replace lim sup by lim inf. Self-conformal measures are Frostman, so there exist $\sigma > 0$ and $0 < \xi_1 < \xi_2 < \cdots \to \infty$ with $|\widehat{\mu}(\xi_n)| \leq \xi_n^{-\sigma}$ for all n.

Example with slow Fourier decay



Example with slow Fourier decay

- Let ν be Cantor–Lebesgue ((1/2, 1/2) measure for IFS $\{T_1(x) = x/3, T_2(x) = (x+2)/3\}$).
- Let

$$F\colon [0,1] \to \mathbb{R}, \quad F(x) := \begin{cases} x + \exp(-\exp(x^{-2})), & x \in (0,1], \\ 0, & x = 0. \end{cases}$$

- Then $F\nu$ is a self-conformal measure for the C^∞ IFS $\{S_1, S_2\} := \{F \circ T_1 \circ F^{-1}, F \circ T_2 \circ F^{-1}\}$, and for all $i_1, \ldots, i_n \in \{1,2\}^n$, $(S_{i_1} \circ \cdots \circ S_{i_n})''(x) \neq 0$ except at finitely many x.
- Moreover

$$\limsup_{\xi \to \infty}((\log \log \xi)|\widehat{F\nu}(\xi)|) > 0.$$

Proof idea

- Let integers $k_1 \leq k_2 \leq \cdots$ be s.t. $3^{k_n} \exp(-\exp(3^{2n})) \approx 0.01$.
- $\widehat{F\nu}(3^{k_n}) = \int_{\mathbb{R}} e^{-2\pi i 3^{k_n} F(x)} d\nu(x) = \int_0^{3^{-n}} \cdots + \int_{2(3^{-n})}^1 \cdots$.
- $\left| \int_0^{3^{-n}} \cdots \right| \approx \nu([0, 3^{-n}]) = 2^{-n}$.
- By methods from Baker–B. ('25), if $\mathcal{F}$ is a family of C^2 maps $f \colon [0,1] \to \mathbb{R}$ with $f''(x) \neq 0 \, \forall f \in \mathcal{F} \, \forall x \in [0,1]$, and $\sup_{f \in \mathcal{F}} \max(||f'||_\infty, ||f''||_\infty) < \infty$, then there is $\sigma > 0$ such that $\forall f \in \mathcal{F}$,
$$|\widehat{f\nu}(\xi)| \lesssim \big(\min_{0 \leq x \leq 1} |f''(x)|\big)^{-1} |\xi|^{-\sigma}.$$
- $\left| \int_{2(3^{-n})}^1 \cdots \right| \leq 2^{-n} \sum_{\underline{i} \in \{1,2\}^n \setminus \{1^n\}} |\widehat{(F \circ T_{\underline{i}})\nu}(3^{k_n})| \ll 2^{-n}$.
- So $|\widehat{F\nu}(3^{k_n})| \approx 2^{-n} \gg (\log\log(3^{k_n}))^{-1}$.

Proof idea for $\limsup_{\xi \to \infty} |\widehat{\mu}(\xi)|/\phi(\xi) > 0$.

- Let
$$W(x) := \begin{cases} e^{-x^{-2}}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$
- Let $c_n \searrow 0$ fast (depending on ϕ), $r_n \searrow 0$ slowly. Define F by $F(0) = 0$, $F'(0) = 1$, and
$$F''(x) := \sum_{n=1}^{\infty} c_n W(x - r_{n+1}).$$
- Then F is C^∞ and $0 < f''(r_{n+1}) \ll f''(r_n)$. Set $\mu := F\nu$ and break up the integral into $[0, r_{n+1}]$, $[r_{n+1}, r_n]$, $[r_n, 1]$.
- To show $(S_{i_1} \circ \cdots \circ S_{i_n})''(x) \neq 0$ except at finitely many x, have to **randomise** choice of c_n and show that a typical choice works.

Recall pushforward result

We will now give a sketch proof of our pushforward result.

Theorem (Baker–B. ('25) / Algom–Chang–Wu–Wu ('25))

Let μ be a self-similar measure on $[0,1]$. Assume $F\colon [0,1] \to \mathbb{R}$ is *either* (i) **real-analytic and non-affine** *or* (ii) C^2 **with** $F''(x) > 0$ $\forall x \in [0,1]$. Then $F\mu$ has polynomial Fourier decay.

Proof sketch

- Disintegrate:

$$\mu = \int_\Omega \mu_\omega dP(\omega).$$

- Each μ_ω is an infinite convolution:

$$\mu_\omega = *_{m=1}^{\infty} \frac{1}{\#[\omega_m]} \sum_{i \in \Delta_{[\omega_m]}} \delta_{t_i \prod_{j=1}^{m-1} r_{[\omega_j]}}.$$

- So $\widehat{\mu_\omega}(\xi) = \prod_{m=1}^{\infty} \frac{1}{\#[\omega_m]} \sum_{i \in \Delta_{[\omega_m]}} \exp\left(-2\pi i \xi t_i \prod_{j=1}^{m-1} r_{[\omega_j]} \right)$.
- Using Erdős–Kahane argument and large deviation theory: $\forall \varepsilon > 0$ we can disintegrate s.t. $\exists \delta > 0$ s.t. $\forall T' > 0$ $\exists \Omega_{T'} \subseteq \Omega$ with $P(\Omega \setminus \Omega_{T'}) \lesssim (T')^{-\delta}$, and $\forall \omega \in \Omega_{T'}$, $\forall T \geq T'$,

$$\mathrm{Leb}(\{\xi \in [-T, T] : |\widehat{\mu_\omega}(\xi)| \geq T^{-\delta}\}) \lesssim T^{\varepsilon}.$$

Proof sketch continued

- Write $\mu_\omega = \mu_{N_\omega} * \lambda_{N_\omega}$ for some N_ω, and Taylor expand F:

$$|\widehat{F\mu_\omega}(\xi)| = \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-2\pi i \xi F(x+y)) d\mu_{N_\omega}(x) d\lambda_{N_\omega}(y) \right|$$
$$\leq \int_{-\infty}^{\infty} \left| \widehat{\mu_{\sigma^{N_\omega}\omega}} \left(\xi F'(x) \prod_{n=1}^{N_\omega} r_{[\omega_n]} \right) \right| d\mu_{N_\omega}(x) + C|\xi|^{-1/3}.$$

- Find s s.t. away from a P-small set of ω, $\mu_\omega((x, x+r)) \leq r^s$.
- Use Erdős–Kahane proposition and nonlinearity assumption: bound the μ_{N_ω}-measure of the x which result in a frequency where $\widehat{\mu_{\sigma^{N_\omega}\omega}}$ is large.
- If *Good* is those ω for which all previous steps work,

$$|\widehat{F\mu}(\xi)| \leq \int_{Good} |\widehat{F\mu_\omega}(\xi)| \, dP(\omega) + P(\Omega \setminus Good) \lesssim |\xi|^{-\delta} + |\xi|^{-\eta}.$$

Thank you for listening!