Distinct dimensions for attractors of iterated function systems

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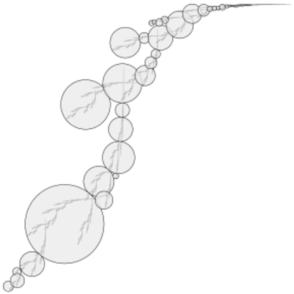
[1]Based on joint work with Simon Baker, De-Jun Feng, Chun-Kit Lai, Ying Xiong https://arxiv.org/abs/2509.22084 and Alex Rutar, https://arxiv.org/abs/2406.12821

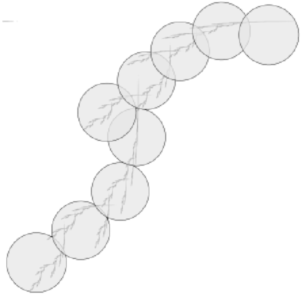
Motivating question

What can properties of a dynamical system tell us about the geometry of invariant sets?

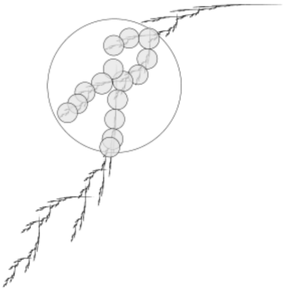
Geometry: coincidence/disparity of fractal dimensions.[2]



Hausdorff



Box/Minkowski



Assouad

[2] Pictures by J. M. Fraser

Fractal dimensions

Sets $F \subset \mathbb{R}^n$ will be non-empty, bounded.

- **Hausdorff** dimension:

$$\dim_{\mathrm{H}} E = \inf\{s \geq 0 : \forall \varepsilon > 0 \, \exists \text{ countable cover } \{U_1, U_2, \ldots\} \text{ of } E \text{ s.t. } \sum_i (\mathrm{diam}(U_i))^s \leq \varepsilon\}.$$

- Lower / upper **box** dimensions:

$$\underline{\dim}_{\mathrm{B}} F = \liminf_{r \to 0} \frac{\log N_r(F)}{\log(1/r)}, \qquad \overline{\dim}_{\mathrm{B}} F = \limsup_{r \to 0} \frac{\log N_r(F)}{\log(1/r)},$$

where $N_r(F)$ is the least number of balls of radius r to cover F.

- **Assouad** dimension:

$$\dim_{\mathrm{A}} F = \inf\{s > 0 : \exists C > 0 \text{ s.t. } \forall 0 < r < R < 1, \forall x \in F, \; N_r(F \cap B(x,R)) \leq C \left(\frac{R}{r}\right)^s\}.$$

Relations between dimensions

- Always
$$\dim_{\mathrm{H}} F \le \underline{\dim}_{\mathrm{B}} F \le \overline{\dim}_{\mathrm{B}} F \le \dim_{\mathrm{A}} F.$$
- If $E = \left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots \right\}$ then
$$\dim_{\mathrm{H}} E = 0 < \frac{1}{2} = \underline{\dim}_{\mathrm{B}} E = \overline{\dim}_{\mathrm{B}} E < 1 = \dim_{\mathrm{A}} E.$$
- If F is those numbers in $[0,1]$ such that for all n, all decimal digits between position 2^{2n} and $(2^{2n+1} - 1)$ are 0, then
$$\dim_{\mathrm{H}} F = \underline{\dim}_{\mathrm{B}} F < \overline{\dim}_{\mathrm{B}} F < \dim_{\mathrm{A}} F.$$

Iterated function systems (IFSs)

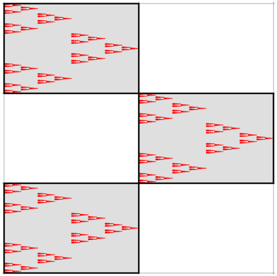
- An IFS is a finite set of contractions $\{S_i \colon X \to X\}_{i \in I}$ (meaning ρ-Lipschitz maps for $\rho < 1$), where $X \subset \mathbb{R}^n$ is compact.
- Hutchinson (1981): there is a unique non-empty compact **attractor/limit set** satisfying

$$F = \bigcup_{i \in I} S_i(F).$$

Thm (Falconer (1989), Feng–Hu (2009))

If all contractions are similarities or C^1 conformal maps (overlaps allowed) then $\dim_{\mathrm{H}} F = \underline{\dim}_{\mathrm{B}} F = \overline{\dim}_{\mathrm{B}} F$.

Bedford–McMullen carpets



Thm (Bedford (1984), McMullen (1984), MacKay (2011))

If F is a Bedford–McMullen carpet without uniform fibres then

$$\dim_{\mathrm{H}} F < \underline{\dim}_{\mathrm{B}} F = \overline{\dim}_{\mathrm{B}} F < \dim_{\mathrm{A}} F.$$

Attractors with distinct dimensions

Question

1. *Does the box dimension of the attractor of an IFS on $\mathbb{R}^d$ always exist?*
2. *Do the Hausdorff and (lower) box dimension of the attractor of an IFS on $\mathbb{R}$ always coincide?*

Thm (Baker–B.–Feng–Lai–Xiong (2025+))

The answer is **no** for both Question 1 and Question 2.

Indeed, there is an IFS of two separated bi-Lipschitz maps on $\mathbb{R}$ such that the attractor F satisfies

$$\dim_{\mathrm{H}} F < \underline{\dim}_{\mathrm{B}} F < \overline{\dim}_{\mathrm{B}} F < \dim_{\mathrm{A}} F.$$

Proof idea

- For every Cantor set $F \subset [0,1]$ containing $\{0,1\}$, $F = S_0(F) \cup S_1(F)$.
- The S_i are bi-Lipschitz contractions iff

$$0 < \inf_{\omega \in \{0,1\}^*} \min\left\{ \frac{|I_{i\omega}|}{|I_\omega|}, \frac{|G_{i\omega}|}{|G_\omega|} \right\} \leq \sup_{\omega \in \{0,1\}^*} \max\left\{ \frac{|I_{i\omega}|}{|I_\omega|}, \frac{|G_{i\omega}|}{|G_\omega|} \right\} < 1.$$

- Symmetric Cantor can give $\underline{\dim}_{\mathrm{B}} F < \overline{\dim}_{\mathrm{B}} F$.
- Asymmetric Cantor set: let $b_0 = 1$, $b_1 = 2$,

$$a_\omega = \frac{b_{\omega_1} \cdots b_{\omega_{\lfloor n/2 \rfloor}}}{\sqrt{b_{\omega_1} \cdots b_{\omega_n}}} \cdot (100)^{-n}.$$

  Separate strings of length $\approx r$ by frequency (p,q) of 1s in 1st/2nd half of coding, and use a lemma of McMullen:

$$\dim_{\mathrm{H}} F = \frac{\log 2}{\log 100} < \underline{\dim}_{\mathrm{B}} F.$$

  'Perturb' by switching between $(100)^{-1}$ and $(101)^{-1}$ to separate upper and lower box dim.

Further questions

Open questions

1. Is $\dim_{\mathrm{H}} F < \underline{\dim}_{\mathrm{B}} F$ or $\underline{\dim}_{\mathrm{B}} F < \overline{\dim}_{\mathrm{B}} F$ possible if the IFS maps are **differentiable**?
2. Does the box dimension of every self-affine set exist?
3. Is there a self-similar set in $\mathbb{R}$ with positive Lebesgue measure but empty interior?

We construct a bi-Lipschitz IFS attractor on $\mathbb{R}$ with positive Lebesgue measure but empty interior.

Infinite conformal IFS (Mauldin–Urbański, 1996)

A **conformal iterated function system** is a countable family of uniformly contracting, $C^{1+\alpha}$ conformal maps $\{S_i \colon X \to X\}_{i \in I}$ on a 'nice' (e.g. non-empty convex compact) set $X \subset \mathbb{R}^d$. We always assume:

- **Open set condition:** $\mathrm{Int}(X) \neq \varnothing$ and $\bigcup_{i \in I} S_i(\mathrm{Int}(X)) \subseteq \mathrm{Int}(X)$ with the union disjoint.
- **Bounded distortion**

The **limit set** is the largest set $F \subseteq X$ (possibly non-compact) satisfying

$$F = \bigcup_{i \in I} S_i(F)$$

Hausdorff and box dimensions

For $w \in I^k$ let R_w be the smallest possible Lipschitz constant for $S_w := S_{w_1} \circ \cdots \circ S_{w_k}$ and define the **pressure function**

$$Pres(t) := \lim_{k \to \infty} \frac{1}{k} \log \sum_{w \in I^k} R_w^t,$$

Theorem (Mauldin–Urbański, 1996, 1999)

- $\dim_{\mathrm{H}} F = h := \inf\{t > 0 : Pres(t) < 0\}$
- $\overline{\dim}_{\mathrm{B}} F = \max\{\dim_{\mathrm{H}} F, \overline{\dim}_{\mathrm{B}} P\}$, where F is obtained by choosing exactly one point from each $S_i(X)$.

Bounds for lower box dimension

Bounds for $\underline{\dim}_{\mathrm{B}} F$ that are immediate from Mauldin–Urbański:

$$\max\{\dim_{\mathrm{H}} F, \underline{\dim}_{\mathrm{B}} P\} \leq \underline{\dim}_{\mathrm{B}} F \leq \overline{\dim}_{\mathrm{B}} F = \max\{\dim_{\mathrm{H}} F, \overline{\dim}_{\mathrm{B}} P\}.$$

Theorem (B.–Rutar, 2024+)

The box dimension of F exists if and only if these bounds coincide.

In fact $\underline{\dim}_{\mathrm{B}} F$ is **not** a function of $\dim_{\mathrm{H}} F$, $\underline{\dim}_{\mathrm{B}} P$, $\overline{\dim}_{\mathrm{B}} P$:

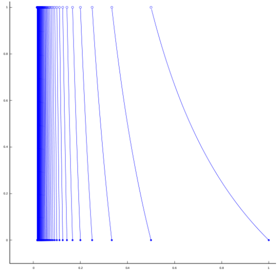
Theorem (B.–Rutar)

The trivial lower bound for $\underline{\dim}_{\mathrm{B}} F$ is sharp, and a sharp upper bound is

$$\underline{\dim}_{\mathrm{B}} F \leq \dim_{\mathrm{H}} F + \frac{(\overline{\dim}_{\mathrm{B}} P - \dim_{\mathrm{H}} F)(d - \dim_{\mathrm{H}} F)\,\underline{\dim}_{\mathrm{B}} P}{d\,\overline{\dim}_{\mathrm{B}} P - \dim_{\mathrm{H}} F\,\underline{\dim}_{\mathrm{B}} P}.$$

Example: continued fraction sets

- Sets F_I with continued fraction entries restricted to $I \subset \mathbb{N}$ are limit sets for CIFS $\{x \mapsto (b+x)^{-1} : b \in I\}$. Can take $P = \{1/b : b \in I\}$.
- They satisfy $\mathcal{G}(F_I) = F_I$ where the Gauss map $\mathcal{G}\colon [0,1) \to [0,1)$ is $\mathcal{G}(x) = \{1/x\}$ and $\mathcal{G}(0) = 0$ (here $\{\cdot\}$ denotes fractional part).[3]



Thm (B.–Rutar, building on Mauldin–Urbański and B.–Fraser)

There exists $I \subset \mathbb{N}$ with $\dim_{\mathrm{H}} F_I < \underline{\dim}_{\mathrm{B}} F_I < \overline{\dim}_{\mathrm{B}} F_I < \dim_{\mathrm{A}} F_I$.

[3] Picture by Adam majewski, CC BY-SA 4.0

Asymptotic formula

- Can derive formula for $\underline{\dim}_{\mathrm{B}} F$ in terms of function
$$r \mapsto s_P(r) := \frac{\log N_r(P)}{\log(1/r)}.$$
-
$$\Psi(r,\theta) := (1-\theta)\dim_{\mathrm{H}} F + \theta s_P(r^\theta),$$
$$\psi(r) := \sup_{\theta \in (0,1]} \Psi(r,\theta).$$
- Write $f(r) \asymp g(r)$ if $f(r) - g(r) \to 0$ as $r \to 0$.

Theorem (B.–Rutar)

If F is the limit set of a CIFS and P is as above then
$$\frac{\log N_r(F)}{\log(1/r)} \asymp \psi(r), \qquad \text{hence} \qquad \underline{\dim}_{\mathrm{B}} F = \liminf_{r \to 0} \psi(r).$$

- The formula can depend on $\dim_{\mathrm{H}} F$, even when $\dim_{\mathrm{H}} F < \underline{\dim}_{\mathrm{B}} P$. It only depends on the contraction ratios via $\dim_{\mathrm{H}} F$.

Proof sketch

- For simplicity assume contractions are similarities and ignore subexponential terms in r.
- Upper bound:

$$N_r(F) \lesssim \sum_{\substack{\omega \in I^* \\ r_\omega > r}} N_{r/r_\omega}(P) \lesssim \sum_{\substack{\omega \in I^* \\ r_\omega > r}} r_\omega^h r^{-\Psi(r,\theta_\omega)} \lesssim r^{-\psi(r)},$$

  where θ_ω is such that $r^{\theta_\omega} = r/r_\omega$.
- Lower bound: extract finite subsystem $\mathcal{F} \subset I$ with $\dim_{\mathrm{H}}$ approximating $\dim_{\mathrm{H}} F$.
  Fix $\theta \in (0,1)$, let $0 < r \ll 1$.

$$\#\{\omega \in \mathcal{F}^* : r_\omega \approx r^{1-\theta}\} \approx (r^{1-\theta})^{-\dim_{\mathrm{H}} F}.$$

  Each such ω contributes $\approx N_{r^\theta}(P) \approx (r^\theta)^{-s_P(r^\theta)}$ to $N_r(F)$.
  So $N_r(F) \gtrsim r^{-\Psi(r,\theta)}$.

Alternative asymptotic formula

- Can reformulate result using order-reversing transformation $x = \log\log(1/r)$. (This sends $[r, r^\theta]$ to $[x - \log(1/\theta), x]$.)
- For $0 \le \lambda \le d$ let $\mathcal{G}(\lambda, d)$ be the set of continuous functions $g\colon \mathbb{R} \to [\lambda, d]$ such that

$$D^+ g(x) \in [\lambda - g(x), d - g(x)],$$

where the Dini derivative is

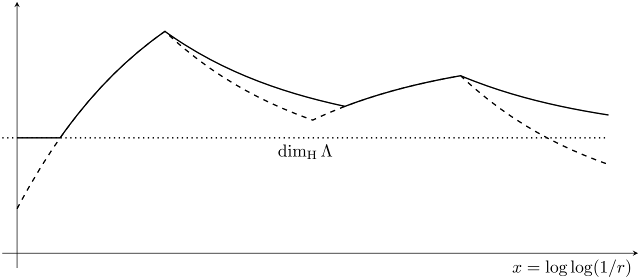
$$D^+ g(x) := \limsup_{\varepsilon \to 0^+} \frac{g(x+\varepsilon) - g(x)}{\varepsilon}.$$

- B.–Rutar (2022) observed that if $E \subset \mathbb{R}^d$ is bounded then $s_E(\exp(\exp(x))) \asymp g(x)$ for some $g \in \mathcal{G}(0, d)$ (we say that E has covering class g), and conversely any $g \in \mathcal{G}(0, d)$ has $s_E(\exp(\exp(x))) \asymp g(x)$ for some E.

Alternative asymptotic formula

Theorem (B.–Rutar)

If P has covering class $f \in \mathcal{G}(0, d)$ and g is the pointwise minimal function $g \geq f$ satisfying $g \in \mathcal{G}(\dim_{\mathrm{H}} F, d)$ then F has covering class g.



'$N_r(F) \geq N_r(P)$ is as small as possible while being at least $\dim_{\mathrm{H}} F$-dimensional between all pairs of scales.'

Dynamical interpretation

Theorem (Barreira (1996), Gatzouras–Peres (1997))

If $f\colon M \to M$ is an expanding, C^1 conformal map of a Riemannian manifold and $F \subseteq M$ is **compact** and invariant (i.e. $f(F) = F$ and $f^{-1}(F) \cap U \subseteq F$ for a neighbourhood U of F), then

$$\underline{\dim}_{\mathrm{H}} F = \underline{\dim}_{\mathrm{B}} F = \dim_{\mathrm{B}} F.$$

Thm (Baker–B.–Feng–Lai–Xiong (2025+))

There is an invariant set F for a **Lipschitz** expanding map on $\mathbb{R}$ with

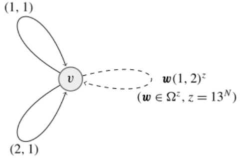
$$\underline{\dim}_{\mathrm{H}} F < \underline{\dim}_{\mathrm{B}} F < \dim_{\mathrm{B}} F.$$

Non-conformal dynamics

Thm (Jurga (2023))

There is a sub-self-affine set ($F \subset \bigcup_i S_i(F)$) with

$$\underline{\dim}_{\mathrm{B}} F < \overline{\dim}_{\mathrm{B}} F.$$



(1, 12)	
(1, 11)	
(1, 10)	
(1, 9)	
(1, 8)	
(1, 7)	
(1, 6)	
(1, 5)	
(1, 4)	
(1, 3)	
(1, 2)	
(1, 1)	(2, 1)

Construction of the set inside a Bedford–McMullen carpet. Picture by N. Jurga.

Thm (Bedford (1984), McMullen (1984), Jurga (2023))

If F is invariant for $(x, y) \mapsto (mx \mod 1, ny \mod 1)$ then $\dim_{\mathrm{H}} F < \underline{\dim}_{\mathrm{B}} F$ or $\underline{\dim}_{\mathrm{B}} F < \overline{\dim}_{\mathrm{B}} F$ are both possible.

Thank you for listening!

谢谢大家