Assouad dimension and its variants in fractal geometry

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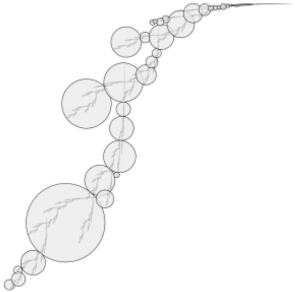
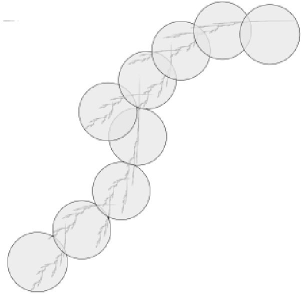
[1]Based on joint work with (various combinations of) Jonathan Fraser, István Kolossváry, Alex Rutar, Sascha Troscheit

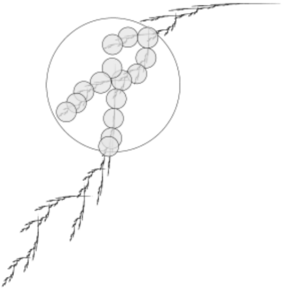
Fractal dimensions



Hausdorff



Box/Minkowski



Assouad

Pictures by Jonathan Fraser

Fractal dimensions

Sets $F \subset \mathbb{R}^n$ will be non-empty, bounded.

- Lower / upper **box** dimensions:

$$\underline{\dim}_{\mathrm{B}} F = \liminf_{r \to 0} \frac{\log N_r(F)}{\log(1/r)}, \qquad \overline{\dim}_{\mathrm{B}} F = \limsup_{r \to 0} \frac{\log N_r(F)}{\log(1/r)},$$

  where $N_r(F)$ is the least number of balls of radius r to cover F.

- **Assouad dimension**:

$$\dim_{\mathrm{A}} F = \inf\{s > 0 : \exists C > 0 \text{ s.t. } \forall 0 < r < R < 1, \forall x \in F,$$
$$N_r(F \cap B(x,R)) \le C \left(\frac{R}{r}\right)^s \}.$$

- **Assouad spectrum** for $\theta \in (0,1)$:

$$\dim_{\mathrm{A}}^{\theta} F = \inf\{s > 0 : \exists C > 0 \text{ s.t. } \forall 0 < R < 1, \forall x \in F,$$
$$N_{R^{1/\theta}}(F \cap B(x,R)) \le C R^{s(1-1/\theta)} \}.$$

- Lü–Xi (2016): **Quasi-Assouad** dimension is $\dim_{\mathrm{qA}} F = \lim_{\theta \to 1^-} \dim_{\mathrm{A}}^{\theta} F$.

Relations between dimensions

- For all θ,

$$\dim_{\mathrm{H}} F \leq \underline{\dim}_{\mathrm{B}} F \leq \overline{\dim}_{\mathrm{B}} F \leq \dim_{\mathrm{A}}^{\theta} F \leq \dim_{\mathrm{qA}} F \leq \dim_{\mathrm{A}} F.$$

- If $E = \left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \right\}$ then

$$\dim_{\mathrm{H}} E = 0 < \frac{1}{2} = \dim_{\mathrm{B}} E < 1 = \dim_{\mathrm{qA}} F = \dim_{\mathrm{A}} E$$

  and $\dim_{\mathrm{A}}^{\theta} F = \min\{\frac{1}{2(1-\theta)}, 1\}$.

- If F is those numbers in $[0, 1]$ such that for all n, all decimal digits between position 2^{2n} and $(2^{2n+1} - 1)$ are 0, then

$$\dim_{\mathrm{H}} F = \underline{\dim}_{\mathrm{B}} F = \frac{1}{3} < \frac{2}{3} = \overline{\dim}_{\mathrm{B}} F < 1 = \dim_{\mathrm{A}} F.$$

Why care?

- Polynomial spiral is

$$S_p := \{x^{-p}e^{ix} : x > 0\}.$$

Chrontsios-Garistis & Tyson (2022): for $a > b > 0$, there is a quasiconformal map f of $\mathbb{C}$ with dilation K_f and $f(S_a) = S_b$ iff $K_f > a/b$.

- Given $F \subset [1,2]$, the spherical maximal function is

$$M_F f = \sup_{t \in F} \left\| \int_{S^{d-1}} f(x - ty) d\sigma(y) \right\|.$$

If $\dim_{\mathrm{A}}^{\theta} F = \dim_{\mathrm{qA}} F$ for $1 - \overline{\dim}_{\mathrm{B}} F / \dim_{\mathrm{qA}} F < \theta < 1$ then Roos & Seeger (2023) calculate the closure of

$$\left\{ \left(\frac{1}{p}, \frac{1}{q} \right) \in [0,1]^2 : M_F \text{ is bounded } L^p \to L^q \right\}.$$

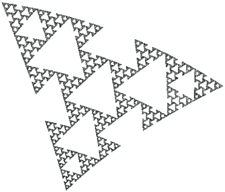
Iterated function systems (IFSs)

- An IFS is a finite set of contractions $\{S_i \colon X \to X\}_{i \in I}$ (meaning ρ-Lipschitz maps for $\rho < 1$), where $X \subset \mathbb{R}^n$ is compact.
- Hutchinson (1981): there is a unique non-empty compact **attractor/limit set** satisfying

$$F = \bigcup_{i \in I} S_i(F).$$

For s-AD-regular sets like self-similar/self-conformal sets (picture by Sabrina Kombrink) with the open set condition,

$$\dim_{\mathrm{H}} F = \overline{\dim}_{\mathrm{B}} F = \dim_{\mathrm{A}} F = s.$$



Bi-Lipschitz and infinite IFSs

Thm (Baker–B.–Feng–Lai–Xiong (2025+))

There is an IFS of two bi-Lipschitz maps on $\mathbb{R}$ such that the attractor F satisfies

$$\dim_{\mathrm{H}} F < \underline{\dim}_{\mathrm{B}} F < \overline{\dim}_{\mathrm{B}} F < \dim_{\mathrm{A}} F.$$

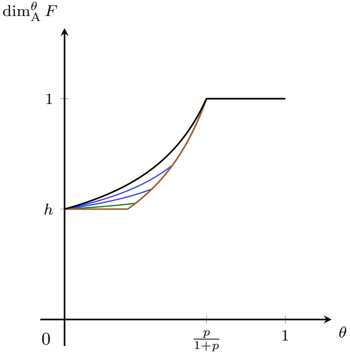
Thm (Mauldin–Urbański (1996,1999), B.–Fraser (2024), B.–Rutar (2024+))

For **infinitely generated** self-conformal sets F,

$$\dim_{\mathrm{H}} F < \underline{\dim}_{\mathrm{B}} F < \overline{\dim}_{\mathrm{B}} F < \dim_{\mathrm{A}} F$$

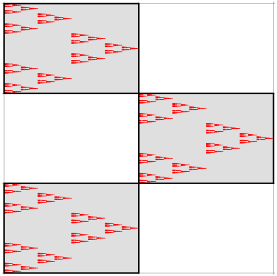
is possible. The Assouad spectrum can have an interesting form with multiple phase transitions.

Infinite IFSs



Bedford–McMullen carpets

$m \times n$ grid, $1 < m < n$, N maps, M non-empty columns with $(N_1, \ldots, N_M)$ maps.



Thm (Bedford (1984), McMullen (1984), Mackay (2011))

If F is a Bedford–McMullen carpet without uniform fibres then

$$\dim_{\mathrm{H}} F < \underline{\dim}_{\mathrm{B}} F = \overline{\dim}_{\mathrm{B}} F < \dim_{\mathrm{A}} F.$$

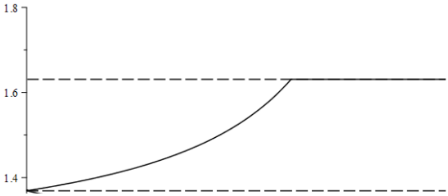
Dimensions of Bedford–McMullen carpets

Thm (Bedford (1984), McMullen (1984), Mackay (2011))

If F is a Bedford–McMullen carpet without uniform fibres then

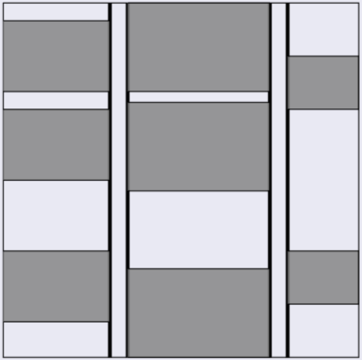
$$\dim_{\mathrm{H}} F < \underline{\dim}_{\mathrm{B}} F = \overline{\dim}_{\mathrm{B}} F < \dim_{\mathrm{A}} F.$$

- $\dim_{\mathrm{B}} F = \frac{\log M}{\log m} + \frac{\log(N/M)}{\log n}$,
- $\dim_{\mathrm{A}} F = \frac{\log M}{\log m} + \max_i \frac{\log N_i}{\log n}$,
- Fraser–Yu (2018):
  $\dim_{\mathrm{A}}^{\theta} F = \dim_{\mathrm{B}} F + \frac{\theta}{1-\theta}\left(\frac{\log n}{\log m} - 1\right)(\dim_{\mathrm{A}} F - \dim_{\mathrm{B}} F)$



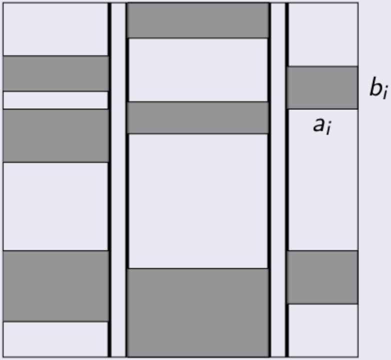
Gatzouras–Lalley carpets (1992)

Homogeneous case



Contractions within columns (in y-direction) is the same.

General case



No condition on contractions except that $b_i < a_i$.

Dimensions of GL carpets

- Vertical projection $\eta(K)$ is self-similar with box dim

$$\sum_{\hat{\jmath}\in\eta(\mathcal{I})} a_{\hat{\jmath}}^{\dim_{\mathrm{B}}\eta(K)} = 1.$$

- Gatzouras–Lalley (1992):

$$\dim_{\mathrm{B}} K = \dim_{\mathrm{B}}\eta(K) + t_{\min} \quad \text{where} \quad \sum_{\hat{\jmath}\in\eta(\mathcal{I})} \sum_{i\in\eta^{-1}(\hat{\jmath})} a_{\hat{\jmath}}^{\dim_{\mathrm{B}}\eta(K)} b_i^{t_{\min}} = 1$$

($t_{\min}$ is weighted "average" column dimension).

- Mackay (2011):

$$\dim_{\mathrm{A}} K = \dim_{\mathrm{B}}\eta(K) + t_{\max} \quad \text{where} \quad t_{\max} = \max_{\hat{\jmath}\in\eta(\mathcal{I})} s_{\hat{\jmath}}; \ \sum_{i\in\eta^{-1}(\hat{\jmath})} b_i^{s_{\hat{\jmath}}} = 1$$

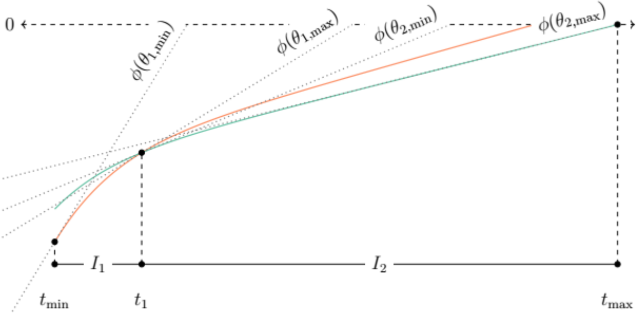
($t_{\max}$ is maximal column dimension).

Column pressure

Let

$$\tau(t) = \begin{cases} \min_{\hat{\jmath} \in \eta(\mathcal{I})} \psi_{\hat{\jmath}}(t) & : t \in [t_{\min}, t_{\max}] \\ -\infty & : \text{ otherwise} \end{cases}$$

where $\psi_{\hat{\jmath}}(t) = \frac{\log \sum_{i \in \eta^{-1}(\hat{\jmath})} b_i^t}{\log a_{\hat{\jmath}}}$.



τ is the minimum of the curves

Parameter change:

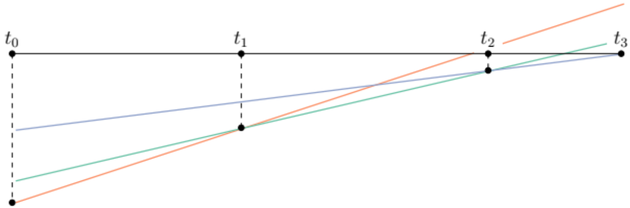
$$\phi(\theta) = \frac{1/\theta - 1}{1 - 1/\kappa_{\max}} \qquad \text{where} \qquad \kappa_{\max} = \max_{i \in \mathcal{I}} \frac{\log b_i}{\log a_i}.$$

Let $\tau^*(\alpha) = \inf_{t \in \mathbb{R}}(t\alpha - \tau(t))$ denote concave conjugate.

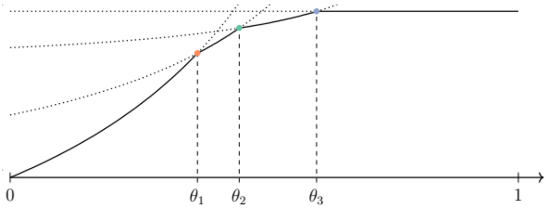
Thm (B.–Fraser–Kolossváry–Rutar (2024+))

$$\dim_{\mathrm{A}}^{\theta} K = \dim_{\mathrm{B}} \eta(K) + \frac{\tau^*(\phi(\theta))}{\phi(\theta)} \qquad \text{for } 0 < \theta < 1.$$

Homogeneous case

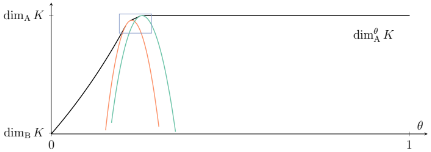


τ is the minimum of the lines

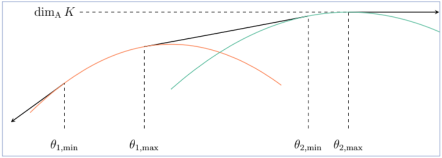


Assouad spectrum can have multiple phase transitions

Non-homogeneous case



(A) Plot of the original spectrum.

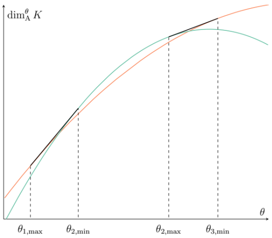


(B) Plot restricted to the rectangular region.

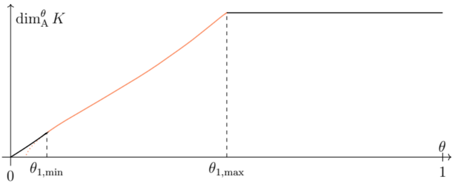
The Assouad spectrum is

- monotonically **increasing**
- **differentiable** because each column is inhomogeneous
- **piecewise analytic** with higher order phase transitions
- has **convex and concave regions**

Strange behaviour



One column can be relevant on many intervals



The "one-column" part is usually concave but can sometimes be convex

Proof idea: variational formula

- Thin cylinders in approx. square: group and cover using $\dim \eta(K)$.
- Thick cylinders: count 'pseudocylinders' using $\dim_{\mathrm{B}} K$ and cover each using $\dim \eta(K)$.
- Covering strategy depends on type class. Size of one class is exponential in n, but only polynomially many classes.
- $$f_{\mathrm{thin}}(\theta, \mathbf{v}, \mathbf{w}) = \dim_{\mathrm{B}} \eta(K) + \frac{H(\mathbf{w}) - H(\eta(\mathbf{w}))}{\chi_2(\mathbf{w})},$$
  $$f_{\mathrm{thick}}(\theta, \mathbf{v}, \mathbf{w}) = \dim_{\mathrm{B}} K + \frac{1}{\phi(\theta, \mathbf{v})} \left(\frac{H(\mathbf{w}) - H(\eta(\mathbf{w})) - t_{\min}\chi_2(\mathbf{w})}{\chi_1(\mathbf{w})} \right),$$
  $$f(\theta, \mathbf{v}, \mathbf{w}) = \begin{cases} f_{\mathrm{thin}}(\theta, \mathbf{v}, \mathbf{w}) & : (\mathbf{v}, \mathbf{w}) \in \Delta_{\mathrm{thin}}(\theta), \\ f_{\mathrm{thick}}(\theta, \mathbf{v}, \mathbf{w}) & : (\mathbf{v}, \mathbf{w}) \in \Delta_{\mathrm{thick}}(\theta). \end{cases}$$

Prop. (BFKR)

$\dim_{\mathrm{A}}^{\theta} K = \max_{(\mathbf{v},\mathbf{w}) \in \mathcal{P} \times \mathcal{P}} f(\theta, \mathbf{v}, \mathbf{w})$.

Proof idea: solving the variational formula

- If $\theta \notin [\theta_{\mathsf{min}}, \theta_{\mathsf{max}}]$ then $\dim_{\mathrm{A}}^{\theta} K$ is the global max of f_{thin} or f_{thick}.
- If $\theta \in [\theta_{\mathsf{min}}, \theta_{\mathsf{max}}]$ then max is attained on the boundary. Constrained optimisation of form $F(\alpha) = \max\{v(\mathbf{w}) : u(\mathbf{w}) = \alpha\}$ formally has an (unconstrained) Lagrange dual $T(t) = \min\{tu(\mathbf{w}) - v(\mathbf{w})\}$, but we can't just apply Lagrange multiplier thm.
- Using that minimisers for T are connected, $F(\alpha) = T^*(\alpha)$. Solve for T.

Recovering the interpolation

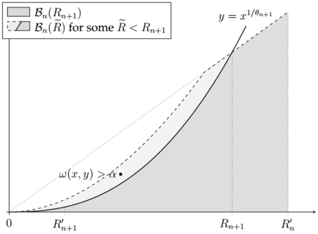
Fraser and Yu suggested defining ϕ-Assouad dimension, $\phi\colon (0,1) \to \mathbb{R}^+$:

$$\dim_{\mathrm{A}}^{\phi} F = \inf\{s > 0 : \exists C > 0 \text{ s.t. } \forall 0 < R < 1, \forall x \in E,$$
$$N_{R^{1+\phi(R)}}(E \cap B(x,R)) \leq CR^{-\phi(R)s}\}.$$

Studied by García–Hare–Mendivil (2021).

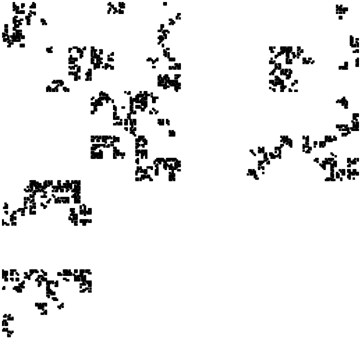
Thm (B.–Rutar–Troscheit (2023+))

For all $F \subset \mathbb{R}^n$ and $\overline{\dim}_{\mathrm{B}} F < s \leq \dim_{\mathrm{A}} F$ there is ϕ_s such that $\dim_{\mathrm{A}}^{\phi_s} F = s$.

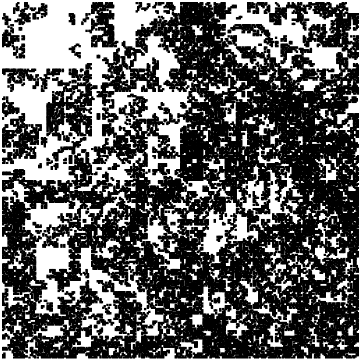


Mandelbrot percolation

$n \times n$ grid in $\mathbb{R}^d$, retain each subcube independently with probability p ad infinitum.



Unconditioned



Conditioned

Dimensions of Mandelbrot percolation

All results are almost sure, conditional on non-extinction:

- For all $\theta \in (0, 1)$,

$$\dim_{\mathrm{H}} M = \dim_{\mathrm{B}} M = \dim_{\mathrm{A}}^{\theta} M = \frac{\log(pn^d)}{\log n}$$

  but $\dim_{\mathrm{A}} M = d$.

- For $\alpha \in [0, \log(n^d)]$, letting

$$\phi_\alpha := \frac{1}{\alpha} \frac{\log\log(1/R)}{\log(1/R)}$$

  we have

$$\dim_{\mathrm{A}}^{\phi_\alpha} M = \alpha \frac{\log(1/p)}{d(\log n)^2} + \frac{\log(pn^d)}{\log n}.$$

Galton–Watson processes

- Mandelbrot perc. result follows from more general one for Galton–Watson processes.
- Let $X_{k,i}$ be i.i.d. random variables with finite support in $\{0, 1, 2, \ldots\}$. Let

$$Z_0 = 1 \qquad \text{and} \qquad Z_{k+1} = \sum_{i=1}^{Z_k} X_{k,i}.$$

Gives a tree with a **Gromov boundary** $\partial\mathcal{T}$.

Large deviations estimate

Assume the $X_{i,k}$ are not almost surely constant, have mean $m > 1$, and their p.g.f is polynomial of degree $N > 2$. Letting γ be s.t. $m^\gamma = N$, for all $1 < t < \gamma$ and small $\varepsilon > 0$ and $k \in \mathbb{N}$,

$$\exp\left(-m^{(t-1+\varepsilon)\frac{\gamma}{\gamma-1}k}\right) \lesssim_{t,\varepsilon} \mathbb{P}\left(Z_k \geq m^{tk}\right) \lesssim_{t,\varepsilon} \exp\left(-m^{(t-1-\varepsilon)\frac{\gamma}{\gamma-1}k}\right).$$

Lemma

Let (E_n) be measurable events for a GW tree. Let $\widetilde{E}$ be the event that there are infinitely many $n \in \mathbb{N}$ s.t. there is a level-n subtree $\mathcal{T}(v) \in E_n$. Then

1. $\mathbb{P}(\widetilde{E}) = 0$ if $\sum_{n\in\mathbb{N}} \mathbb{P}(E_n) m^n < \infty$,
2. $\mathbb{P}(\widetilde{E}) = 1$, conditioned on non-extinction, if there is a summable sequence $K_n \geq 0$ s.t. $\sum_{n\in\mathbb{N}} K_n \mathbb{P}(E_n) m^n = \infty$.

Combining with large deviations estimate gives:

Thm (B.–Rutar–Troscheit)

$$\dim_{\mathrm{A}}^{\phi_\alpha} \partial\mathcal{T} = \alpha \left(1 - \frac{\log m}{\log N}\right) + \log m.$$

Overlapping self-similar sets

Thm (Fraser–Henderson–Olson–Robinson (2015))

If self-similar $F \subset \mathbb{R}$ satisfies weak separation then $\dim_{\mathrm{A}} F = \dim_{\mathrm{H}} F$, otherwise $\dim_{\mathrm{A}} F = 1$.

Conjecture

If $F \subset \mathbb{R}$ is self-similar then $\dim_{\mathrm{A}}^{\theta} F = \dim_{\mathrm{H}} F$ for all $\theta \in (0,1)$.

This conjecture is true under exponential separation (Fraser–Yu (2018) using Shmerkin (2019)).

Open problem

Exhibit a self-similar $F \subset \mathbb{R}$ with $\dim_{\mathrm{H}} F < \dim_{\mathrm{A}} F = 1$ and functions ϕ_s such that $\dim_{\mathrm{A}}^{\phi_s} F = s$ for $s \in (\dim_{\mathrm{H}} F, 1]$.

Thank you for listening!

谢谢大家