FOURIER TRANSFORM OF NONLINEAR IMAGES OF SELF-SIMILAR MEASURES: QUALITATIVE ASPECTS

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Abstract. The goal of this paper is to establish polynomial Fourier decay for images of self-similar measures μ on $\mathbb{R}^k$ under sufficiently nonlinear real-analytic maps $f\colon \mathbb{R}^k \to \mathbb{R}^d$. For example, we prove that if f is analytic on $\mathbb{R}^k$, its graph does not lie in a proper affine subspace of $\mathbb{R}^{k+d}$, and μ is not supported in a proper affine subspace of $\mathbb{R}^k$, then the image measure has polynomial Fourier decay. Key steps in the proof include establishing a uniform Łojasiewicz-type inequality for self-similar measures, and using the decay of the Fourier transform of μ outside a very small exceptional set of frequencies. As an application of our results, we prove polynomial Fourier decay for self-conformal measures on $\mathbb{C}$ for a large class of complex analytic IFSs which are not self-similar but are conjugate to linear via an analytic map.

Contents

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1. Introduction

1.1. **Fourier decay of nonlinear images.** In this paper, we consider polynomial Fourier decay for non-linear images of self-similar measures. The problem is given as follows: let k, d be positive integers and let $f\colon \mathbb{R}^k \to \mathbb{R}^d$ be a real analytic map. Let μ be a self-similar measure on $\mathbb{R}^k$, and consider the pushforward measure μ_f on $\mathbb{R}^d$. We ask the following question about its Fourier transform; as we will see, the answer will depend on properties of μ and f.

Question 1.1. *Does μ_f have polynomial Fourier decay? In other words, are there some $\sigma, c > 0$ such that*

$$|\widehat{\mu_f}(\boldsymbol{\xi})| \leqslant c|\boldsymbol{\xi}|^{-\sigma} \qquad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^d \setminus \{\mathbf{0}\}?$$

Remark 1.2. *If μ is the Lebesgue measure on a bounded open set in $\mathbb{R}^k$ (rather than a fractal measure), then the polynomial Fourier decay property for μ_f holds if f is non-degenerate (our definition of non-degeneracy will be made precise in Section 2.1). This is a classical result in harmonic analysis, see [56, VIII, Section 3.2].*

Question 1.1 is motivated by work of Kaufman [32], who observed that the image measure can have polynomial Fourier decay even if the self-similar measure is a non-Rajchman measure (meaning that $\limsup_{|\boldsymbol{\xi}|\to\infty} |\widehat{\mu}(\boldsymbol{\xi})| > 0$) like the Cantor–Lebesgue measure; we defer further historical background to a later section. Our answer to Question 1.1 is given by the following theorem, which is the main result of the paper. To avoid trivialities, we make the standing assumption throughout the paper that all our self-similar measures are never just a single atom.

Theorem 1.3. *Let μ be a self-similar measure on $\mathbb{R}^k$ for some integer $k \geqslant 1$. Let $f\colon \mathbb{R}^k \to \mathbb{R}^d$ be a real analytic map where $d \geqslant 1$ is an integer, and assume $f(\mathbb{R}^k)$ is not contained in a proper affine hyperplane of $\mathbb{R}^d$. Then μ_f has polynomial Fourier decay unless all of the following conditions hold:*

(1) μ is supported in an affine hyperplane $H \subseteq \mathbb{R}^k$ (possibly all of $\mathbb{R}^k$), and the restriction $f|_H$ is partially linear in the sense that there are real numbers, $b_1, \ldots, b_d$, not all zero, such that for $f = (f_1, \ldots, f_d)$, the function $\sum_i b_i f_i$ restricts to an affine map $H \to \mathbb{R}$.

(2) *Given H as above, if ν is a self-similar measure on $\mathbb{R}^{\dim H}$ and $\iota\colon \mathbb{R}^{\dim H} \hookrightarrow H$ is an isometric embedding then ν does* not *have polynomial Fourier decay.*[1]

We will see in Section 2.1 that the partial linearity conclusion from Theorem 1.3 (1) is equivalent to the graph of $f|_H$ not being contained in an affine subspace of $H \times \mathbb{R}^d$ which is proper in the sense that its dimension is strictly less than $d + \dim H$.

The proof of Theorem 1.3 contains substantial additional difficulties compared to previous work in the $d = 1$ or $k = 1$ cases. In particular, we need to establish a uniform Łojasiewicz type inequality for self-similar measures, see Section 3.3. Moreover, we need to use methods of Khalil [33] to prove that self-similar measures on $\mathbb{R}^k$ which are not supported in a proper affine subspace of $\mathbb{R}^k$ have polynomial Fourier decay outside a very sparse set of exceptional frequencies, see Section 3.4.

Theorem 1.3 is sharp, as illustrated by the following simple observation.

Proposition 1.4. *Let $f\colon \mathbb{R}^k \to \mathbb{R}^d$ be an analytic map which is partially linear on $\mathbb{R}^k$. Then there is some self-similar measure μ on $\mathbb{R}^k$ which is not supported in any proper affine subspace of $\mathbb{R}^k$, and such that neither μ nor μ_f is Rajchman.*

Proof. There exist $b_1, \dots, b_d$ such that $\sum_i b_i f_i$ is a linear function. Next, observe that

$$\widehat{\mu_f}(\boldsymbol{\xi}) = \int e^{-2\pi i \sum_i \xi_i f_i(x)} d\mu(x).$$

Then if $\boldsymbol{\xi}$ is chosen to be along the line with direction $(b_1, \dots, b_d)$ we see that $\sum_i \xi_i f_i(x) = t_{\boldsymbol{\xi}} L(x)$ for some non-trivial[2] linear form L and $t_{\boldsymbol{\xi}}$ is a real number whose norm is proportional to the length of $\boldsymbol{\xi}$. We write this linear form as

$$L(x) = \sum_{i=1}^{k} a_i x_i + b$$

for real numbers $a_1, \dots, a_k, b$. Then observe that for any self-similar measure μ on $\mathbb{R}^k$,

$$\int e^{-2\pi i \sum_i \xi_i f_i(x)} d\mu(x) = \left| \int e^{-2\pi i t_{\boldsymbol{\xi}} \sum_i a_i x_i} d\mu(x) \right| = |\widehat{\mu}(t_{\boldsymbol{\xi}}(a_1, \dots, a_k))|. \tag{1.1}$$

[1] Whether ν has polynomial Fourier decay is clearly independent of the choice of ν and ι.

[2] Otherwise the non-decay property is trivial.

Let μ be the k-fold product of the Cantor–Lebesgue measure, rotated so that $\widehat{\mu}$ does decay along the direction $(a_1, \ldots, a_k)$. Then from (1.1) we see that $\widehat{\mu_f}$ does not decay along $(b_1, \ldots, b_d)$. □

Proposition 1.4 gives a converse to Theorem 1.3, because if we let $H \subseteq \mathbb{R}^k$ be the smallest affine subspace for which (1) and (2) hold, then Proposition 1.4 tells us that $\mu_f = \nu_{f_H \circ \iota}$ does not have polynomial Fourier decay.

By Theorem 1.3 and Proposition 1.4, the problem of characterising polynomial Fourier decay for images of self-similar measures on $\mathbb{R}^k$ by maps $\mathbb{R}^k \to \mathbb{R}^d$ which are analytic on $\mathbb{R}^k$ has been reduced to the problem of characterising those self-similar measures on $\mathbb{R}, \mathbb{R}^2, \ldots, \mathbb{R}^k$ which have polynomial Fourier decay. It must be noted, however, that this latter problem is extremely challenging and remains wide open even for Bernoulli convolutions, which are a special class of self-similar measures on $\mathbb{R}$.

One of the applications of our results relates to Fourier decay for self-conformal measures in the plane. One headline result is the following. We will contextualise the results and describe precisely what we mean by all the terms in the following statements in Section 7.

Theorem 1.5. *Let ν be a self-conformal measure for an IFS of holomorphic contractions on a ball in $\mathbb{C}$. Assume that ν is not self-similar and not supported inside a finite union of analytic curves. Then ν has polynomial Fourier decay.*

In fact, we prove Theorem 1.5 in the case when the conformal IFS can be conjugated to a self-similar IFS by a holomorphic diffeomorphism. The non-conjugate case is a direct consequence of a recent result of Algom, Rodriguez Hertz and Wang [5].

Structure of the paper. Section 2 gives necessary preliminaries and references to relevant background literature. In Section 3 we state Theorem 3.1, which is a stronger result than Theorem 1.3 as it does not require the domain of the analytic map to be the whole of $\mathbb{R}^k$. Theorem 3.1 shows that various combinations of conditions for the self-similar measure and analytic map guarantee polynomial Fourier decay for the pushforward. Theorem 3.3 considers Fourier decay of lifts of self-similar measures to graphs of nonlinear maps.

Section 4 deduces Theorems 3.1 and 3.3 from Lemma 4.1 and two additional ingredients, namely Theorems 3.6 and 3.9. Lemma 4.1 approximates the Fourier transform of the image measure by an average of the Fourier transform of pieces of the self-similar measure across many frequencies; this was already proved as Lemmas 3.2 and 3.3 in [11].

Theorems 3.6 establishes a Łojasiewicz type inequality for self-similar measures and is proved in Section 5. Theorem 3.9 establishes Fourier decay of self-similar measures outside a sparse set of frequencies and is proved in Section 6. Section 7 applies our results to self-conformal measures, proving Theorem 1.5 using Theorem 3.1. Finally, Section 8 presents several possible avenues for future research. See Figure 1 for a schematic representation of the paper.

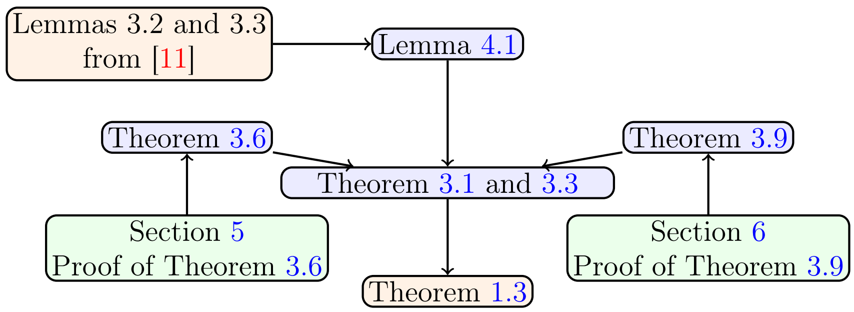

FIGURE 1. Structure of the paper.

2. Preliminaries and background literature

2.1. **Real analytic maps and their properties.** We explain some terminologies that we use. Let $k, d \geqslant 1$ be integers.

Definition 2.1. *Given a non-empty set $K \subseteq \mathbb{R}^k$, we say that $f = (f_1, \ldots, f_d)\colon K \to \mathbb{R}^d$ is* (real) analytic *if there exists some open neighbourhood $U \subseteq \mathbb{R}^k$ of K such that for all $\mathbf{x} = (x_1, \ldots, x_k) \in U$ and $1 \leqslant j \leqslant d$, there is some power series in $x_1, \ldots, x_k$ which converges to $f_j(\mathbf{x})$ in some open neighbourhood of $\mathbf{x}$.*

Now let $U \subseteq \mathbb{R}^k$ be open, and let $f\colon U \to \mathbb{R}^d$ be analytic. Let $\mathbf{v} \in \mathbb{S}^{d-1}$. Define $f_{\mathbf{v}}\colon U \to \mathbb{R}$ by $f_{\mathbf{v}}(\mathbf{x}) \coloneqq \sum_{j=1}^{d} v_j f_j(\mathbf{x})$. Define

$$P_{\mathbf{v}}\colon U \to \mathbb{R}^k, \qquad P_{\mathbf{v}}(\mathbf{x}) \coloneqq \nabla f_{\mathbf{v}}(\mathbf{x}) = (\nabla f(\mathbf{x}))^T(\mathbf{v}).$$

We define several different assumptions on our function f that we will use.

Definition 2.2. *Let $f\colon U \to \mathbb{R}^d$ be analytic.*

1. *We say that f is* non-degenerate *if for all balls $B \subset \mathbb{R}^k$ and all $\mathbf{v} \in \mathbb{S}^{d-1}$, $P_{\mathbf{v}}$ is not a constant function on $B \cap U$.*
2. *We say that f is* non-conical *if for all balls $B \subset \mathbb{R}^k$ and all $\mathbf{v} \in \mathbb{S}^{d-1}$, $\mathbf{x} \mapsto |P_{\mathbf{v}}(\mathbf{x})|$ is not a constant function on $B \cap U$.*

(3) *We say that f is* non-trapped *if for all balls $B \subset \mathbb{R}^k$, $f(B \cap U)$ is not contained in a proper affine subspace of $\mathbb{R}^d$.*

We make the following observations about the relations between these conditions.

Remark 2.3. (1) *If f is non-conical then f is clearly non-degenerate, and the two conditions are equivalent when $k = 1$.*
(2) *If f is non-degenerate then f is clearly non-affine and non-trapped.*
(3) *The map $f\colon \mathbb{R}^k \setminus \{\mathbf{0}\} \to \mathbb{R}$ ($k > 1$), $\mathbf{x} \mapsto |\mathbf{x}|$ (whose graph is a cone) is an example of a non-degenerate but conical function.*
(4) *The map $f\colon \mathbb{R} \to \mathbb{R}^2$, $x \mapsto (x, x^2)$ is an example of a non-affine and non-trapped analytic map that is degenerate.*

The following proposition shows that our non-degeneracy is equivalent to a more familiar notion of non-degeneracy for analytic functions, see Kleinbock and Margulis [34].

Proposition 2.4. *Given an analytic map $f\colon U \to \mathbb{R}^d$, the following are equivalent:*
(1) *f is degenerate,*
(2) *f is partially linear on some ball $B \subset \mathbb{R}^k$ in the sense that there are $b_1, \ldots, b_d \in \mathbb{R}$, not all 0, such that $\sum_i b_i f_i$ restricts to an affine map $B \cap U \to \mathbb{R}$.*
(3) *For some ball $B \subset \mathbb{R}^k$ the graph of $f_{B\cap U}$ is contained in a proper affine subspace of $\mathbb{R}^{k+d}$.*

Proof. (1) is equivalent to the existence of a ball B and $\mathbf{v} \in \mathbb{S}^{d-1}$ such that $P_{\mathbf{v}}$ is constant on $B \cap U$ (or equivalently $f_{\mathbf{v}}$ is affine on $B \cap U$). This implies that $(v_1 f_1 + \cdots + v_d f_d)(\mathbf{x}) = L(\mathbf{x})$ on $B \cap U$ for some affine form L, which gives (2). Now (2) implies that $(\mathbf{x}, f_1, \ldots, f_d)$ satisfies some non-trivial affine form on $B \cap U$, which is equivalent to (3).

Conversely, (3) implies that there are numbers $a_1, \ldots, a_k, b_1, \ldots, b_d, b$, not all 0, such that

$$\sum_{i=1}^{k} a_i x_i + \sum_{i=1}^{d} b_i f_i(\mathbf{x}) + b = 0$$

for all $\mathbf{x} = (x_1, \ldots, x_k) \in B \cap U$. Not all the b_i can be zero otherwise we would have a linear condition which cannot be satisfied by all values of the free variables $x_1, \ldots, x_k$. Therefore we see (2) holds. Now (2) implies that $P_{(b_1,\ldots,b_d)}$ is constant on $B \cap U$, giving (1). □

Indeed, if $P_{\mathbf{v}}$ is a constant function for some $\mathbf{v} \in \mathbb{S}^{d-1}$, then $f_{\mathbf{v}}$ is affine. This says that $(v_1 f_1 + \cdots + v_d f_d)(\mathbf{x}) = L(\mathbf{x})$ for some affine linear form L. This implies that $(\mathbf{x}, f_1, \dots, f_d)$ satisfies some non-trivial linear form and therefore the graph of f in $\mathbb{R}^{k+d}$ is contained in some affine hyperplane.

Note that our example $\mathbf{x} \mapsto |\mathbf{x}|$ for a non-degenerate conical function is not globally analytic on $\mathbb{R}^k$. Indeed, the following result holds.

Proposition 2.5. *Let $f\colon \mathbb{R}^k \to \mathbb{R}^d$ be analytic on $\mathbb{R}^k$. Then f is conical if and only if f degenerates.*

Proof. We only need to prove the forward implication. Suppose $P_{\mathbf{v}}$ is constant for some $\mathbf{v}$. Then $|\nabla f_{\mathbf{v}}|$ is constant. It is well known from the theory of Eikonal equation that $f_{\mathbf{v}}$ must be affine, see for example [59]. But $f_{\mathbf{v}}$ being affine implies that $P_{\mathbf{v}}\colon \mathbf{x} \mapsto \nabla f_{\mathbf{v}}(\mathbf{x})$ is a constant map, hence the degeneracy. □

For our uniform measure Łojasiewicz inequality we will need the following definition.

Definition 2.6. *Let $K \subseteq \mathbb{R}^k$ and consider a family $\mathcal{F}$ of analytic maps from $K \to \mathbb{R}^d$. We say that $\mathcal{F}$ is* compact *if it is compact in the compact-open topology, in other words every sequence of functions in $\mathcal{F}$ contains a subsequence which converges uniformly on compact subsets of K to an element of $\mathcal{F}$.*

2.2. **Self-similar sets and measures.** For general fractal geometry background we refer the reader to the books [13, 23, 38, 39]. Let $k \geqslant 1$ and $N > 1$ be integers, and let $D \subset \mathbb{R}^k$ be compact. Let $\Lambda = \{f_i\colon D \to D\}_{1 \leqslant i \leqslant N}$ be contraction maps (i.e. ρ-Lipschitz maps for some $\rho < 1$); Λ is called an *iterated function system* or *IFS* for short. Let $p_1, \dots, p_N \in (0,1)$ be such that $\sum_i p_i = 1$. By Hutchinson's theorem [28], there is a unique non-empty compact set $K \subset D$, called the *attractor*, and a unique Borel probability measure μ whose support equals K, such that

$$K = \bigcup_i f_i(K), \qquad \mu = \sum_i p_i f_i(\mu).$$

We will always assume that the contractions do not share a common fixed point, ensuring that K is uncountable and μ is non-atomic.

If there exist $r_1, \dots, r_N \in (0,1)$ and $O_1, \dots, O_N \in O_k(\mathbb{R})$ (possibly reflected) rotations and $\mathbf{t}_1, \dots, \mathbf{t}_N \in \mathbb{R}^k$ such that Λ consists of the similarity maps

$$\Lambda = \{f_i(\cdot) = r_i O_i(\cdot) + \mathbf{t}_i\}_{i \in \{1, \dots, N\}},$$

then we say that Λ, K, μ are self-similar. In this case, we say that Λ, μ, K are homogeneous if $r_i O_i$ are all the same for $i \in \{1, \dots, N\}$.

Now let μ be a self-similar measure on $\mathbb{R}^k$. Without loss of generality, we can assume that the support of μ is contained in a unit cube. Let Λ be an IFS for μ. For each $\omega \in \Lambda^*$ (the space of finite-length sequences with symbols in Λ), following the composition of maps $\omega_0\omega_1\cdots$, there is a least $l \geqslant 0$ such that the contraction ratio corresponding to $\omega_0^l = \omega_0\cdots\omega_l$ is at most $1/2^n$. We write $l = l_{\omega,n}$. Notice that the finite collection of paths

$$\{\omega_0^{l_{\omega,n}}\}_{\omega\in\Lambda}$$

corresponds to a covering of $\mathrm{supp}(\mu)$. Such a covering uses similar copies of $\mathrm{supp}(\mu)$ of sizes at most $1/2^n$ and at least $\rho_m 2^{-n}$ where ρ_m is smallest contraction ratio for μ. Let $\mu_{\omega,n}$ be the corresponding similar copy of μ with probability weight $|\mu_{\omega,n}|$. Clearly, each such branch $\mathrm{supp}(\mu_{\omega,n})$ intersects at least one and at most 2^k many dyadic cubes in $\mathcal{D}_n$. We choose only one such dyadic cube and associate it with $\mu_{\omega,n}$. As a result, for each dyadic cube $D_n \in \mathcal{D}_n$, there is a collection of similar copies $\mu_{\omega,n}$ that are associated with this cube. We write $\mu_{\omega,n} \sim D_n$ for this association. We will call such a collection of ω to be Λ_{D_n}. We then obtain the following decomposition of Λ

$$\{\Lambda_{D_n}\}_{D_n\in\mathcal{D}_n}$$

which induces a decomposition (or disintegration) of μ. For each $\omega \in \Lambda_{D_n}$, the copy $\mu_{\omega,n}$ is supported in $3D_n$, the tripling of D_n with the same centre.

2.3. Affinely irreducibility.

Definition 2.7. *We say that a Borel probability measure μ on $\mathbb{R}^k$ is affinely irreducible if for any proper affine subspace $H \subset \mathbb{R}^k$, $\mu(H) = 0$.*

For example, if $k = 1$, then any self-similar measure whose support is not a finite set must be affinely irreducible. As another example, notice that normalised Lebesgue measures on bounded open sets are affinely irreducible. This example extends to natural surface carried measures on non-degenerate manifolds.

2.4. Expanding systems.

Definition 2.8. *Let G be a group. Let $F \subset G$ be a finite set. We say that G is* non-expanding *with respect to F (or F is* non-expanding*) if for each $\varepsilon > 0$,*

$$\#F_n \ll e^{\varepsilon n},$$

where $F_n \subset G$ is the collection of $f_1 \cdots f_n$ for $f_1, \ldots, f_n \in F$.

The Tits alternative [45,58] gives a more algebraic description of the expanding property.

Definition 2.9 (non-expanding self-similar system)**.** *Let* Λ *be a self-similar IFS. We say that it is* non-expanding *if the collection of linear parts* $\{O_i\}$ *is non-expanding viewed as a subset of the Euclidean group on* $\mathbb{R}^k$*. We also say that a self-similar set/measure is non-expanding if one of its corresponding self-similar IFS is non-expanding.*

For example, any self-similar system with abelian rotation group is non-expanding. In particular, every self-similar system in $\mathbb{R}$ or $\mathbb{R}^2$ is non-expanding.

2.5. **Fourier decay of non-linear images of self-similar measures.** The study of non-linear images of self-similar measures was initiated by Kaufman [32] for Bernoulli convolutions on the real line and later extended by [43,44,60] to homogeneous self-similar measures. Finally, the problem for $f\colon \mathbb{R} \to \mathbb{R}$ was completed for general self-similar measures by [1,8]. Recently, in [11] we quantified (explicitly lower-bounded the decay exponent of) all of the previous results and moreover considered the more general case $f\colon \mathbb{R}^k \to \mathbb{R}$. For vector valued functions, [6] considered the case $f\colon \mathbb{R} \to \mathbb{R}^d$ where $d \geqslant 1$ can be arbitrary. The input of this paper is to provide a complete answer to the Fourier decay problem for all analytic $f\colon \mathbb{R}^k \to \mathbb{R}^d$ where $k, d \geqslant 1$ are arbitrary. There are still some 'boundary cases' that we do not cover when the domain of f is not the whole of $\mathbb{R}^k$, but these cases are rather special, see Section 8.

2.6. **Self-conformal measures.** Self-conformal measures are a natural generalisation of self-similar measures. Given a set $D \subset \mathbb{R}^k$, we call a map $f\colon D \to \mathbb{R}^k$ *conformal* if on some open neighbourhood U of D it is $C^{1+\alpha}$ and preserves the angle and orientation of directed curves through each point in U. Equivalently, it is $C^{1+\alpha}$ on U and the Jacobian at each point is a positive scalar multiplied by a rotation matrix with determinant 1. When $k = 2$, conformal maps are precisely holomorphic maps with a non-vanishing complex derivative at every point in U. When $k \geqslant 3$, conformal maps have a very restricted form by Liouville's theorem and are in fact Möbius transformations: they can be written as a composition of translations, rotations, reflections, similarities, and inversions in $(k-1)$-spheres. Note in particular that when $k \geqslant 2$, conformal maps are real analytic.

Next, we introduce self-conformal measures. For simplicity, we do not attempt to give the most general possible setup. Let D be the closure of a non-empty, convex, open set in $\mathbb{R}^k$, and assume that there is a bounded, open, convex set $U \subseteq \mathbb{R}^k$ with $D \subset U$. Let $\phi_1, \ldots, \phi_N \colon U \to \mathbb{R}^k$ be injective conformal maps satisfying $\overline{\varphi_i(U)} \subset U$ (where $\overline{\varphi_i(U)}$

denotes the topological closure of $\varphi_i(U)$) and $\phi_i(D) \subset D$ for each i. Assume moreover that

$$0 < \inf\{||(D\phi_i)_z|| : z \in U, 1 \leqslant i \leqslant N\} \\ \leqslant \sup\{||(D\phi_i)_z|| : z \in U, 1 \leqslant i \leqslant N\} < 1.$$

Assume that $\phi_1, \ldots, \phi_N$ do not preserve a common fixed point. We call $\{\phi_i, \ldots, \phi_N\}$ a *conformal IFS*. By [28], there is a unique non-empty compact set $K \subset D$ called the *self-conformal set* associated with this system, and a unique Borel probability measure μ called the *self-conformal measure*, such that

$$K = \bigcup_i f_i(K), \qquad \mu = \sum_i p_i \phi_i(\mu).$$

Note that K is uncountable, μ is non-atomic, and the support of μ equals K. In this paper, whenever we talk about self-conformal IFSs, sets or measures, we will mean those which arise in this way, and we will mostly be interested in the $k = 2$ case.

For measures on the line (i.e. $k = 1$), Fourier decay results for nonlinear fractal measures (in particular self-conformal measures) has received a great deal of attention, see for instance [2, 3, 10, 15, 29, 31, 35, 49, 52] and the survey [51]. It is known from a combination of recent results that self-conformal measures for an IFS of analytic contractions $\mathbb{R} \to \mathbb{R}$ which are not all affine have polynomial Fourier decay [1, 4, 8, 12]. In the $k = 2$ case, polynomial Fourier decay of self-conformal measures which are not conjugate to linear has been studied in [5] (see Section 7.1 for more details). Polynomial Fourier decay for self-conformal measures in $\mathbb{R}^k$ for general k has been considered in [10], under the strong separation condition and a uniform non-integrability assumption.

3. Proof ideas and further results

3.1. **More refined pushforward results.**

In the introduction, we assumed the pushforward maps were analytic on the whole of $\mathbb{R}^k$. However, we can also prove several results for maps which are only analytic on an open neighbourhood of the support of μ. We will deduce Theorem 1.3 easily from Theorem 3.1 in Section 3.2.

Theorem 3.1. *Let $k, d \geqslant 1$ be integers, let μ be an affinely irreducible self-similar measure on $\mathbb{R}^k$, let $U \subseteq \mathbb{R}^k$ be an open set containing* $\mathrm{supp}(\mu)$*, and let $f \colon U \to \mathbb{R}^d$ be real analytic. Assume that at least one of the following three conditions holds:*

(1) f is non-conical, or

(2) μ is non-expanding and f is non-degenerate, or

(3) μ has polynomial Fourier decay and f is non-trapped.

Then μ_f has polynomial Fourier decay.

We next make some comments about how sharp Theorem 3.1 is.

Remark 3.2. *(1) We are not presently able to prove that the conclusion of Theorem 3.1 holds when the non-conicality assumption (1) is replaced by non-degeneracy (without further assumptions on μ). This is precisely [11, Conjecture 5.14] – Theorem 3.1 makes very good progress towards this conjecture, for example Theorem 3.1 (2) resolves it affirmatively in the case $k \in \{1, 2\}$.*

(2) The non-degeneracy assumption in (2) cannot be relaxed, because if f were degenerate then by the proof of Proposition 1.4 there would be some non-Rajchman self-similar measure whose pushforward is non-Rajchman.

(3) The non-trapped assumption in (3) cannot be relaxed, because if $f(B \cap U)$ were contained in some subspace of $\mathbb{R}^d$ (for some ball B) then the image of any self-similar measure whose support intersects $f(B \cap U)$ would not have Fourier decay along a direction orthogonal to the subspace.

Regarding Theorem 3.1 (3), we note that while some self-similar measures (such as the Cantor–Lebesgue measure) are non-Rajchman, in the line self-similar measures ‘typically’ have polynomial Fourier decay in some very strong sense [55], and countably many specific examples with polynomial Fourier decay are known [18, 57].

Our method proving Theorems 3.1 can be used to show that lifts of self-similar measures to the graph of a nonlinear function have Fourier decay in all nontrivial directions with a uniform exponent.

Theorem 3.3. *Let $k, d \geqslant 1$ be integers and let μ be a affinely irreducible self-similar measure on $\mathbb{R}^k$. Let $U \subseteq \mathbb{R}^k$ be an open set containing $\operatorname{supp}(\mu)$, and let $f\colon U \to \mathbb{R}^d$ be analytic. Assume that at least one of the following conditions holds:*

(1) f is not the sum of a conical function and an affine function, or

(2) μ is non-expanding and f is non-degenerate.

Define $T\colon U \to \mathbb{R}^{k+d}$ by $\mathbf{x} \mapsto (\mathbf{x}, f(\mathbf{x}))$. Then there exists $\sigma > 0$ such that for all $\boldsymbol{\xi} \in \mathbb{R}^{k+d}$ of unit length, with the last d coordinates of $\boldsymbol{\xi}$ not all zero, there exists $c_{\boldsymbol{\xi}} > 0$ such that for all $t > 0$,

$$\widehat{\mu_T(t\boldsymbol{\xi})} \leqslant c_{\boldsymbol{\xi}} t^{-\sigma}.$$

Remark 3.4. *The $k = 1$ case of Theorem 3.3 was established via a different method in [6, Proposition 1.5], see also Remark 3.7 (1).*

Proposition 3.5. *Let $k, d \geqslant 1$ be integers and let μ be a self-similar measure on $\mathbb{R}^k$ with polynomial Fourier decay. Let $U \subseteq \mathbb{R}^k$ be an open set containing $\operatorname{supp}(\mu)$, and let $f \colon U \to \mathbb{R}^d$ be analytic and non-degenerate. Define $T \colon U \to \mathbb{R}^{k+d}$ by $\mathbf{x} \mapsto (\mathbf{x}, f(\mathbf{x}))$. Then μ_T has polynomial Fourier decay.*

3.2. Deducing Theorem 1.3.

Theorem 1.3 is a direct consequence of Theorems 3.1 (1) and (3).

Proof of Theorem 1.3 using Theorem 3.1. Let H be the smallest affine subspace containing the support of μ. We restrict f to H and regard H as the ambient space; then we can assume that μ is affinely irreducible. From Theorem 3.1 (1) we can deduce the desired Fourier decay property unless f is conical. Since f is globally real-analytic, recall from Proposition 2.5 that conicality is equivalent to degeneracy, which by Proposition 2.4 is equivalent to partial linearity. This finishes the proof of (1). Now (2) follows immediately from Theorem 3.1 (3). □

The proof of Theorem 3.1 is the most technical part of this paper. We first illustrate the ingredients which are of independent interest.

3.3. A uniform measure Łojasiewicz inequality.

Our next result, Theorem 3.6, is a non-concentration result which generalises a classical result of Łojasiewicz [37] which is well-known in real algebraic/analytic geometry [16]. Being unable to find an existing version of the Łojasiewicz inequality that suits our needs, we will provide in this paper a standalone proof of the following result. Such a result may have other applications in future. Recently, in a paper on Diophantine approximation, Bénard, He and Zhang used related results [14, Theorem 4.4 and Corollary 4.5] (for polynomials rather than general analytic functions) to prove a Khintchine-type dichotomy for self-similar measures in $\mathbb{R}^k$.

Theorem 3.6. *Let $\mathcal{A}$ be a compact family of real analytic functions from some compact $K \subset \mathbb{R}^k$ to $\mathbb{R}^d$. Let μ be an affinely irreducible self-similar measure on $\mathbb{R}^k$. Assume that $\mathcal{A}$ does not contain the zero function. Then for all $\varepsilon > 0$, there are numbers $c, \delta > 0$ such that for all $f \in \mathcal{A}$, $r > 0$,*

$$\mu(\{|f| < r\}^{r^{\varepsilon}} \cap K) \leqslant c r^{\delta}.$$

Remark 3.7.

(1) If $k = 1$, then we have one variable real analytic functions. In this case, standard complex analysis can be used to establish this theorem. Namely, for $k = 1$, Theorems 3.1 and 3.3 can be proved via a simpler method.

(2) *Even in the* $d = 1$ *case, our proof of Theorems 1.3 and 3.1 contain substantial additional ingredients compared to the* $d = 1$ *results in [11]. This is because in [11] we assumed that the graph of the pushforward function has nonvanishing Gaussian curvature, which means that the zero sets of the functions in* $\mathcal{A}$ *in our application of Theorem 3.6 consist of isolated points, so the proof of Theorem 3.6 simplifies substantially.*

The requirement that $\mathcal{A}$ be an analytic compact family on a compact set K allows the consideration of analytic functions that are only defined around K rather than globally.

If $\mathcal{A}$ is a finite collection and μ is the Lebesgue measure on any bounded open set. Then the above result can be deduced from the classical Łojasiewicz inequality which states that for some $\delta > 0$ and all small enough $r > 0$,

$$\{|f| < r\} \subset \{f = 0\}^{r^\delta}.$$

Since f is not constant, $\{f = 0\}$ is a proper analytic variety. Thus it is not difficult to show that $\mu(\{f = 0\}^{r^\delta}) \ll r^{\delta''}$ for some $\delta'' > 0$. This shows Theorem 3.6 in this specific case. In fact, it is not difficult to extend this argument to all affinely irreducible self-similar measures rather than the Lebesgue measure. The point of Theorem 3.6 is that the collection of functions may not be finite. Therefore, some further steps are needed to achieve uniformity.

3.4. **Fourier decay outside sparse frequencies.** We will also need the fact that affinely irreducible self-similar measures have Fourier decay outside of a very sparse set of frequencies. More precisely, we have the definition of the following property, which can be shown to be equivalent to an L^2-*flattening* property (increase in L^2 dimension of iterated convolutions, as in [6, (1.3)] for instance).

Definition 3.8. *Let* μ *be a probability measure on* $\mathbb{R}^k$*. We say that* μ *(or* $\widehat{\mu}$*) has the* Fourier decay outside sparse frequencies *property if for each* $\varepsilon > 0$ *there exist* $\delta, C > 0$ *so that*

$$|\{\boldsymbol{\xi} \in \mathbb{R}^k : |\widehat{\mu}(\boldsymbol{\xi})| \geqslant R^{-\delta}, |\boldsymbol{\xi}| < R\}| \leqslant CR^\varepsilon$$

for all $R > 0$*, where* $|\cdot|$ *denotes Lebesgue measure on* $\mathbb{R}^k$*.*

All measures with polynomial Fourier decay have Fourier decay outside sparse frequencies, but so do many non-Rajchman measures such as the Cantor–Lebesgue measure. The following result holds.

Theorem 3.9. *A self-similar measure on* $\mathbb{R}^k$ *has Fourier decay outside sparse frequencies if and only if it is affinely irreducible.*

There has been a significant amount of prior work showing different classes of fractal measures having decay outside sparse frequencies. It was verified for Bernoulli convolutions by Kaufman [32] using an Erdős–Kahane argument, for self-similar measures in the line (this is the $k = 1$ case of Theorem 3.9) by Tsujii [60], certain homogeneous self-similar measures in arbitrary dimensions by Mosquera and Olivo [44, Proposition 2.2 and Section 5], and certain infinitely generated self-similar measures by Baker and Banaji [8, Corollary 4.11]. One could attempt to quantify the dependence between δ and ε as done by Mosquera and Shmerkin in [43], but that is beyond the scope of this paper.

In [33, Corollary 1.8], motivated by proving exponential mixing of geodesic flows, Khalil proved that every measure (not necessarily self-similar) which satisfies a (local) uniform affine non-concentration condition (or in fact a weaker non-concentration condition from [33, Definition 11.1 and Corollary 11.4]) has Fourier decay outside sparse frequencies. It is not difficult to see that for self-similar measures, uniform affine non-concentration holds under the strong separation condition but not in general. The forward implication of Theorem 3.9 is straightforward. To establish the backward implication for (possibly overlapping) self-similar measures we introduce a weaker non-concentration condition which we call inner-affine non-concentration (see Definition 6.1), prove that affinely irreducible self-similar measures satisfy this condition (see Theorem 6.2), and observe that Khalil's proof of decay outside sparse frequencies goes through under this condition (see Theorem 6.4).

As well as its use in the proofs of Theorem 3.1 (1) and (2), Theorem 3.9 is also used in the proof of [10, Theorem 1.5] which gives polynomial Fourier decay for certain inhomogeneous self-similar measures.

3.5. **Further applications.** In addition to Theorem 1.5 on self-conformal measures, there are several further applications of polynomial Fourier decay which can be deduced immediately by combining results from the literature with results in this paper. We briefly describe a couple of these.

(1) *Normality and effective equidistribution:* We say that a point $\mathbf{x} \in \mathbb{R}^d$ is *normal* if for every expanding integer matrix $A \in M_{d\times d}(\mathbb{Z})$, $(A^n\mathbf{x})_{n=1}^{\infty}$ is equidistributed in the torus $\mathbb{R}^d/\mathbb{Z}^d$ when reduced mod 1. If the Fourier transform of a probability measure μ decays fast enough (polynomial decay suffices) then μ-a.e. point is normal; see [10, Theorem A.1] (attributed to Fraser and Sahlsten). The $d = 1$ case was proved earlier in [21, 48],

and in this case one can deduce stronger quantitative equidistribution results, see for instance [48, Theorems 1 and 3]. One can use [10, Theorem A.1] to deduce, for example, that if μ is Cantor–Lebesgue measure then $(x^2, x^3, x^4, \dots, x^{d+1}) \in \mathbb{R}^d$ is normal for μ-a.e. x.

(2) *Fourier restriction on fractals:* One can deduce Fourier restriction estimates for measures which have polynomial Fourier decay, such as those from Theorems 1.3 and 1.5. Indeed, from work such as [42, Theorem 4.1], [40, Corollary 3.1] and [56, page 353] it is known that if a measure μ on $\mathbb{R}^k$ has polynomial Fourier decay then there is some $p_\mu > 1$ such that for all $p \in [1, p_\mu]$ the Fourier transform can be thought of as a bounded linear operator $L^p(\mathbb{R}^k, \text{Lebesgue}) \to L^2(\text{supp}(\mu), \mu)$.

4. Proving Theorems 3.1 and 3.3 assuming Theorem 3.6

In the proofs of Theorem 3.1 (1) and (2) below we will also assume Theorem 3.9. Given a self-similar measure μ and a smooth map $f\colon U \to \mathbb{R}^d$ where $U \supset \text{supp}(\mu)$ is open, write $\Gamma_f = \{(\mathbf{x}, f(\mathbf{x})) : \mathbf{x} \in U\} \subset \mathbb{R}^{k+d}$ for the graph of f. Given $\mu_{\omega,n} \sim D_n \subset U$ and $\boldsymbol{\xi} \in \mathbb{R}^{k+d}$ with first k coordinates 0, let $\boldsymbol{\xi}_{\omega,n} \in \mathbb{R}^k$ be such that $(\boldsymbol{\xi}_{\omega,n}, \mathbf{0})$ is the projection of $\boldsymbol{\xi}$ to $\mathbb{R}^k \times \mathbf{0} \subset \mathbb{R}^{k+d}$ along the direction that is orthogonal to $T_{\omega,n}$, i.e.

$$((\boldsymbol{\xi}_{\omega,n}, \mathbf{0}) - \boldsymbol{\xi}) \perp T_{\omega,n}. \tag{4.1}$$

We first use the following result from our previous paper.

Lemma 4.1. *[Lemmas 3.2 and 3.3 from [11]] Let μ be a self-similar measure on $\mathbb{R}^k$, let $f\colon \mathbb{R}^k \to \mathbb{R}^d$ be C^2, and let $T(\mathbf{x}) = (\mathbf{x}, f(\mathbf{x}))$. Then for sufficiently large $n \in \mathbb{N}$ and $\boldsymbol{\xi} \in \mathbb{R}^{k+d}$ with first k coordinates being 0,*

$$|\widehat{\mu_f}(\xi_{k+1}, \dots, \xi_{k+d})| = |\widehat{\mu_T}(\boldsymbol{\xi})| \leqslant \sum_{D_n} \sum_{\mu_{\omega,n} \sim D_n} \left|\widehat{\mu}_{\omega,n}(\boldsymbol{\xi}_{\omega,n})\right| + O_{\mu,f}(|\boldsymbol{\xi}|/2^{2n}).$$

4.1. **Non-expanding self-similar measures.** We first show Theorem 3.1 (2). Let U be an open set containing the support of μ on which f is real analytic.

For each $\mathbf{v} \in \mathbb{S}^{d-1}$ we defined $f_\mathbf{v}$ to be the function given by the inner product $\mathbf{x} \mapsto \langle f(\mathbf{x}), \mathbf{v}\rangle$. Next, for $\xi > 0$, we define $P_{\xi\mathbf{v}}\colon \mathbf{x} \mapsto (\nabla f_\mathbf{v}(\mathbf{x}))^T(\xi) \in \mathbb{R}^k$. We see that $P_\mathbf{v}$ is real analytic. For each $\mathbf{y} \in \mathbb{R}^k$, consider the set

$$E_{\mathbf{v},\mathbf{y}} \coloneqq P_\mathbf{v}^{-1}(\{\mathbf{y}\}). \tag{4.2}$$

We see that $E_{\mathbf{v},\mathbf{y}}$ is a (not necessarily proper/non-singular) analytic subvariety of $\mathbb{R}^k$ defined by the analytic equation $\{\mathbf{x} : P_\mathbf{v}(\mathbf{x}) - \mathbf{y} = \mathbf{0}\}$.

The correspondence between varieties and equations is fixed in this way. We denote this collection of subvarieties by $\mathcal{M}_f$.

If for some $\mathbf{v}, \mathbf{y}$ the subvariety $E_{\mathbf{v},\mathbf{y}}$ has dimension k, then $E_{\mathbf{v},\mathbf{y}}$ is the whole of $\mathbb{R}^k$. In this case, we see that $P_{\mathbf{v}}$ is a constant map. This implies that $\nabla f_{\mathbf{v}}$ is a constant map. This implies that the graph of $f_{\mathbf{v}}$ is always normal to a fixed vector. Thus $f_{\mathbf{v}}$ is an affine function. This again implies that f is degenerate.

Next, we consider the case that f is non-degenerate, so the $E_{\mathbf{v},\mathbf{y}}$ never have dimension k, and are always proper analytic subvarieties of $\mathbb{R}^k$.

Lemma 4.2. *Let μ be a non-expanding self-similar measure on $\mathbb{R}^k$ with Fourier decay outside sparse frequencies, and let U be an open neighbourhood of* $\operatorname{supp}(\mu)$. *Let $f \colon U \to \mathbb{R}^d$ be real analytic and non-degenerate. Suppose μ uniformly decays near $\mathcal{M}_f$. Then μ_f has polynomial Fourier decay.*

Here, 'μ uniformly decays near $\mathcal{M}_f$' means that for each $\varepsilon > 0$ there exist $c = c(\varepsilon)$, $\eta = \eta(\varepsilon) > 0$ such that for all $\mathbf{v} \in \mathbb{S}^{d-1}$, $\mathbf{y} \in \mathbb{R}^k$, $m \in \mathcal{M}_f$, $\delta > 0$,

$$\mu(\{|f_m| \leqslant \delta\}^{\delta^{\varepsilon}}) \leqslant c\delta^{\eta},$$

where f_m is the defining analytic function for m.

Proof of Lemma 4.2. We start with Lemma 4.1,

$$|\widehat{\mu_T}(\boldsymbol{\xi})| \leqslant \sum_{D_n} \sum_{\mu_{\omega,n} \sim D_n} \left|\widehat{\mu}_{\omega,n}(\boldsymbol{\xi}_{\omega,n})\right| + O(|\boldsymbol{\xi}|/2^{2n}). \tag{4.3}$$

For each scaled and rotated copy $\mu_{\omega,n}$ of μ, we have $\widehat{\mu}_{\omega,n}(\boldsymbol{\xi}_{\omega,n}) = |\mu_{\omega,n}|\widehat{\mu}(r_{\omega,n}O_{\omega,n}(\boldsymbol{\xi}_{\omega,n}))$ for some scaling $r_{\omega,n} \asymp 2^{-n}$ and rotation $O_{\omega,n} \in O_k(\mathbb{R})$. It is convenient to define L_n to be the following set of probability weights and scaling ratios of $\mu_{\omega,n}$:

$$L_n = \{(p, r, R) : p, r, R \text{ are the probability weight, scaling ratio and rotation for some } \mu_{\omega,n}\}.$$

Thus, L_n offers a way to classify branches $\mu_{\omega,n}$. We rewrite (4.3) according to this classification,

$$\widehat{\mu_T}(\boldsymbol{\xi}) \leqslant \sum_{g \in L_n} \sum_{D_n \in \mathcal{D}_n} \sum_{\substack{\mu_{\omega,n} \in g \\ \mu_{\omega,n} \sim D_n}} |\mu_{\omega,n}||\widehat{\mu}(r_{\omega,n}O_{\omega,n}(\boldsymbol{\xi}_{\omega,n}))| + O(|\boldsymbol{\xi}|/2^{2n}).$$

For $\mu_{\omega,n} \sim g$, we write $2^{-\kappa_g n} = |\mu_{\omega,n}|$ for the probability weight of those $\mu_{\omega,n}$. We now define $\mathcal{C}_n$ to be the collection of all D_0 (dyadic cubes of unit scale) that intersects $\{r_{\omega,n}O_{\omega,n}(\boldsymbol{\xi}_{\omega,n})\}_{D_n \in \mathcal{D}_n}$. For each $g \in L_n$, we write $\mu_{\omega,n} \sim g$ if it has the indicated (by g) probability weight and scaling ratio. Notice that $\# L_n \ll 2^{\varepsilon n}$ for each $\varepsilon > 0$ because of the non-expanding property. Therefore

$$|\widehat{\mu_T}(\boldsymbol{\xi})| \ll \sum_{D_0 \in \mathcal{C}_n} \sum_{g \in L_n} \sum_{\mu_{\omega,n} \in g r_{\omega,n} O_{\omega,n} \boldsymbol{\xi}_{\omega,n} \in D_0} |\mu_{\omega,n}||\widehat{\mu}(r_{\omega,n} O_{\omega,n}(\boldsymbol{\xi}_{\omega,n}))| + |\boldsymbol{\xi}|/2^{2n}$$
$$\ll \sum_{g \in L_n} \frac{1}{2^{\kappa_g n}} \sum_{D_0 \in \mathcal{C}_n} \sum_{\substack{\mu_{\omega,n} \in g \\ r_{\omega,n} O_{\omega,n} \boldsymbol{\xi}_{\omega,n} \in D_0}} |\widehat{\mu}(r_{\omega,n} O_{\omega,n}(\boldsymbol{\xi}_{\omega,n}))| + |\boldsymbol{\xi}|/2^{2n}.$$

Write $O_{\omega,n}$ for the rotation part of $\mu_{\omega,n}$; it is fixed according to g. From here we write

$$N_{D_0}(g) \coloneqq \#\{\mu_{\omega,n} : \mu_{\omega,n} \in g, r_{\omega,n} O_{\omega,n}(\boldsymbol{\xi}_{\omega,n}) \in D_0\}.$$

Then we see that

$$|\widehat{\mu_T}(\boldsymbol{\xi})| \ll \sum_{g \in L_n} \frac{1}{2^{\kappa_g n}} \sum_{D_0 \in \mathcal{C}_n} N_{D_0}(g) \int_{D_0} |\widehat{\mu}(\boldsymbol{\xi}')| d\boldsymbol{\xi}' + |\boldsymbol{\xi}|/2^{2n}. \tag{4.4}$$

Since μ has Fourier decay outside sparse frequencies, for each $\varepsilon > 0$ we can find a $\delta > 0$ so that

$$\sum_{\substack{D_0 : \exists \boldsymbol{\xi}' \in D_0, \\ |\widehat{\mu}(\boldsymbol{\xi}')| \geqslant |\boldsymbol{\xi}/2^n|^{-\delta}}} 1 \ll 2^{\varepsilon n}.$$

We then have

$$\sum_{g \in L_n} \frac{1}{2^{\kappa_g n}} \sum_{D_0 \in \mathcal{C}_n} N_{D_0}(g) \int_{D_0} |\widehat{\mu}(\boldsymbol{\xi}')| d\boldsymbol{\xi}'$$
$$\ll \sum_{g \in L_n} \frac{1}{2^{\kappa_g n}} \sum_{D_0 \in \mathcal{C}_n} N_{D_0}(g) |\boldsymbol{\xi}/2^n|^{-\delta}$$
$$+ \sum_{g \in L_n} \frac{1}{2^{\kappa_g n}} \sum_{\substack{D_0 : \exists \boldsymbol{\xi}' \in D_0, \\ |\widehat{\mu}(\boldsymbol{\xi}')| \geqslant |\boldsymbol{\xi}/2^n|^{-\delta}}} N_{D_0}(g) + O(|\boldsymbol{\xi}|/2^{2n}).$$

Observe that because μ is a probability measure,

$$\sum_{g \in L_n} \frac{1}{2^{\kappa_g n}} \sum_{D_0 \in \mathcal{C}_n} N_{D_0}(g) \ll \sum_{g \in L_n} \frac{1}{2^{\kappa_g n}} \#\{\mu_{\omega,n} : \mu_{\omega,n} \in g\} \ll 1.$$

On the other hand (because of Fourier decay outside sparse frequencies) we have

$$\sum_{g \in L_n} \frac{1}{2^{\kappa_g n}} \sum_{\substack{D_0 : \exists \boldsymbol{\xi}' \in D_0, \\ |\widehat{\mu}(\boldsymbol{\xi}')| \geqslant |\boldsymbol{\xi}/2^n|^{-\delta}}} N_{D_0}(g) \ll \sum_{g \in L_n} \frac{1}{2^{\kappa_g n}} 2^{\varepsilon n} \max_{D_0} N_{D_0}(g).$$

Claim 4.3. *For some $\eta > 0$ so that uniformly for all D_0 in consideration,*

$$N_{D_0}(g) = \#\{\mu_{\omega,n} : \mu_{\omega,n} \in g, \boldsymbol{\xi}_{\omega,n} \in r_g^{-1} R_g^{-1}(D_0)\} \ll \left(\frac{2^n}{|\boldsymbol{\xi}|}\right)^{\eta} 2^{\kappa_g n}. \tag{4.5}$$

Here r_g is the scaling ratio indicated by g and R_g is the rotation indicated by g.

From this claim, we obtain

$$\sum_{g \in L_n} \frac{1}{2^{\kappa_g n}} \sum_{\substack{D_0 : \exists \boldsymbol{\xi}' \in D_0, \\ |\widehat{\mu}(\boldsymbol{\xi}')| \geqslant |\boldsymbol{\xi}/2^n|^{-\delta}}} N_{D_0}(g) \ll 2^{2\varepsilon n} \left(\frac{2^n}{|\boldsymbol{\xi}|}\right)^{\eta}.$$

Given the claim, (4.4) gives

$$|\widehat{\mu_T}(\boldsymbol{\xi})| \ll |\boldsymbol{\xi}/2^n|^{-\delta} + 2^{2\varepsilon n}(2^n/|\boldsymbol{\xi}|)^{\eta} + |\boldsymbol{\xi}|/2^{2n}. \tag{4.6}$$

We choose $|\boldsymbol{\xi}| \asymp 2^{1.5n}$ and this implies that

$$|\widehat{\mu_T}(\boldsymbol{\xi})| \ll |\boldsymbol{\xi}|^{-\sigma}$$

for some $\sigma > 0$, as required. Since the above holds for all n and all choice of $\boldsymbol{\xi}$ with $|\boldsymbol{\xi}| \asymp 2^{1.5n}$, this finishes the proof. □

Proof of Claim 4.3. We need the decay property for μ near $\mathcal{M}_f$, which we assumed to hold. The idea is that the support of each $\mu_{\omega,n} \in g$ so that $\boldsymbol{\xi}_{\omega,n} \in r_g^{-1} R_g^{-1}(D_0)$ is contained in a thin neighbourhood of an analytic submanifold of $\mathbb{R}^k$. More precisely, recall the map $P_{\xi\mathbf{v}} \colon \mathbb{R}^k \to \mathbb{R}^k$ defined by

$$P_{\xi\mathbf{v}} \colon \mathbf{x} \in \mathbb{R}^k \mapsto (\nabla f(\mathbf{x}))^T(\xi\mathbf{v}) = (\nabla f_{\mathbf{v}}(\mathbf{x}))^T(\xi)$$

where $\mathbf{v} \in \mathbb{S}^{d-1}$, and $\xi \geqslant 0$ is such that $\boldsymbol{\xi} = \xi\mathbf{v}$, and $(\nabla f(\mathbf{x}))^T \colon \mathbb{R}^d \to \mathbb{R}^k$ denotes the transpose of the linear map describing by the derivative of f at $\mathbf{x}$. This map has the property that for $\mathbf{x}_0 \in \mathbb{R}^k$ being the center of D_n as in the beginning of this proof, and $\boldsymbol{\xi} = \xi\mathbf{v}$, the corresponding $\boldsymbol{\xi}_{\omega,n}$ is precisely $P_{\xi\mathbf{v}}(\mathbf{x}_0)$. Thus, the branches $\mu_{\omega,n}$ so that $\boldsymbol{\xi}_{\omega,n} \in R_g^{-1}(D_0)$ is located in the inverse image $P_{\xi\mathbf{v}}^{-1}(D_0')$ for some cube D_0' of size $\asymp 1/r_g \asymp 2^n$, that is to say, $P_{\mathbf{v}}^{-1}(D'')$ for some cube D'' of size $\asymp 2^n/\xi$.

Now, we can consider $E_{\mathbf{v},\mathbf{y}}$ for a suitable $\mathbf{y} \in \mathbb{R}^k$ (the center of D''). More precisely, $E_{\mathbf{v},\mathbf{y}}$ is the variety defined via the analytic function $f_{\mathbf{v},\mathbf{y}} \colon \mathbf{x} \to P_{\mathbf{v}}(\mathbf{x}) - \mathbf{y}$. In this case, by the uniform decay property, we see that for each chosen $\varepsilon > 0$, $\mu(\{|P_{\mathbf{v}}(\mathbf{x}) - \mathbf{y}| < \delta\}^{\delta^{\varepsilon}}) \leqslant c\delta^{\eta}$ for some constants c, η and all $\delta > 0$. By choosing ε suitably small[3], we see that the total μ mass of all above discussed branches $\mu_{\omega,n}$ is $\ll (2^n/\xi)^\eta$. This implies that the number of such branches is $\ll (2^n/\xi)^\eta 2^{\kappa_g n}$. This proves the claim. □

Proof of Theorem 3.1 under hypothesis (2)*, assuming Theorems 3.6, 3.9.* All is left to verify the hypothesis of Lemma 4.2. Decay outside sparse frequencies follows from the affine irreducibility of μ and Theorem 3.9. It remains to verify the uniform decay property. By Theorem 3.6, we need to verify that the collection of analytic functions $P_{\mathbf{v}}(\cdot) - \mathbf{y}$ is a compact family excluding the zero function. By uniformly bounding $|\nabla f|$ over the compact set $\mathrm{supp}(\mu)$, we see that if $|\mathbf{y}|$ is sufficiently large then for all $\mathbf{v} \in \mathbb{S}^{d-1}$, $\mathrm{supp}(\mu) \cap E_{\mathbf{v},\mathbf{y}} = \varnothing$. Therefore it suffices to consider $\mathbf{v}, \mathbf{y}$ which range over compact sets, so the compactness of the family follows. The non-containment of the zero function follows from the non-degeneracy of f. □

4.2. **Expanding self-similar measures.** The proof of Theorem 3.1 under the assumption (1) is similar to the proof under assumption (2). If $P_{\mathbf{v}}$ is not constant, then for each $r \in [0, \infty)$, the set

$$E_{\mathbf{v},r} = \{|P_{\mathbf{v}}(\mathbf{x})| = r\}$$

is an analytic variety. Its dimension cannot be k, otherwise $|P_{\mathbf{v}}|^2$ (which is analytic) would be a constant function. Those subvarieties form a collection $\mathcal{M}_f^K$ (K for "cone"). For such an $E_{\mathbf{v},r}$, we assign the defining equation to be

$$|P_{\mathbf{v}}(\mathbf{x})|^2 = r^2.$$

This is because $\mathbf{x} \mapsto |P_{\mathbf{v}}(\mathbf{x})|^2 - r^2$ is real analytic. We have the following result.

Lemma 4.4. *Let μ be a self-similar measure on $\mathbb{R}^k$ with Fourier decay outside sparse frequencies and let U an open neighbourhood of* $\mathrm{supp}(\mu)$*. Let $f\colon U \to \mathbb{R}^d$ be real analytic and assume that for every $\mathbf{v} \in \mathbb{S}^{d-1}$, $|P_{\mathbf{v}}|$ is non-constant on each connected component of U. Suppose μ uniformly decays near $\mathcal{M}_f^K$, then μ_f has polynomial Fourier decay.*

Proof. The proof is very similar to that of Lemma 4.2. We need to rectify the difficulty that our self-similar system may now have a large rotation group.

[3]Here ε should be small to make sure that for $\delta \asymp 2^n/\xi$, δ^ε should be $\gg 1/2^n$. For $\xi \asymp 1/2^{1.5n}$ as required in (4.6), $\varepsilon = 0.0001$ would be sufficient.

We can follow the proof of Lemma 4.2 until the construction of L_n. Because the rotation group is not non-expanding we must not include it in L_n. The new L_n is defined as

$$L_n = \{(p, r) : p, r \text{ is the probability weight and scaling ratio for some } \mu_{\omega,n}\}.$$

Notice that $\#L_n$ grows only polynomially with respect to n. We can then follow the rest of the proof of Lemma 4.2 until we hit the construction of $N_{D_0}(g)$. It is now

$$N_{D_0}(g) = \#\{\mu_{\omega,n} : \mu_{\omega,n} \in g, O_{\omega,n}(\boldsymbol{\xi}_{\omega,n}) r_{\omega,n} \in D_0\}.$$

We can no longer control the rotation part $O_{\omega,n}$. Nonetheless, we claim that

$$\begin{aligned} N_{D_0}(g) &= \#\{\mu_{\omega,n} : \mu_{\omega,n} \in g, O_{\omega,n}(\boldsymbol{\xi}_{\omega,n}) r_{\omega,n} \in D_0\} && (4.7)\\ &\ll \#\{\mu_{\omega,n} : \mu_{\omega,n} \in g, \boldsymbol{\xi}_{\omega,n} \in A_{D_0/r_g}\} && (4.8)\\ &\ll \left(\frac{2^n}{|\boldsymbol{\xi}|}\right)^{\eta} 2^{\kappa_g n}. && (4.9)\end{aligned}$$

Here r_g is the scaling ratio indicated by g and A_{D_0/r_g} is the set of points $\boldsymbol{\xi}$ in $\mathbb{R}^k$ so that $R(\boldsymbol{\xi}) \in D_0/r_g$ for some rotation R. Thus A_{D_0/r_g} is an annulus of thickness $\asymp 1/r_g \asymp 2^n$. This annulus can have an inner radius 0.

Thus, the effect of not controlling the rotation part is that instead of considering preimages of cubes under $P_{\xi\mathbf{v}}$, we need to consider preimages of annuli (which are much larger than cubes[4]) under $P_{\xi\mathbf{v}}$. The condition that $|P_{\mathbf{v}}|$ is not constant for all $\mathbf{v}$ and the fact that μ uniformly decays near $\mathcal{M}_f^K$ prove the claim. This is similar to the last part of the proof of Lemma 4.2. This finishes the proof. $\square$

Proof of Theorem 3.1 under hypothesis (1)*, assuming Theorems 3.6, 3.9.* We again verify the assumptions of Lemma 4.2. Decay outside sparse frequencies follows from Theorem 3.9. Uniformly bounding $|\nabla f|$ over $\operatorname{supp}(\mu)$ we can take $\mathbf{v}, r$ to lie in a compact region and obtain a compact family $\{|P_{\mathbf{v}}(\cdot)|^2 - r^2\}_{\mathbf{v},r}$. Here, $|P_{\mathbf{v}}(\cdot)|^2$ is the square sum of components of $P_{\mathbf{v}}(\cdot)$, which is real analytic. This family does not contain the zero function since f is non-conical. Using Theorem 3.6 then gives the decay property and finishes the proof. $\square$

[4]Cubes are in some sense zero-dimensional objects, while annuli are $k-1$ dimensional objects.

4.3. **Self-similar measures with polynomial Fourier decay.** We can also use similar methods to prove Theorem 3.1 under assumption (3).

Proof of Theorem 3.1 (3). The forward implication is clear, since if a piece of μ_f is contained in a line then μ_f has no Fourier decay along the direction orthogonal to that line. For the backward implication, assume the image of each connected component of the domain of f which intersects U is not contained in any proper affine subspace of $\mathbb{R}^d$. Our assumptions on f mean that for $\mathbf{v} \in \mathbb{S}^{d-1}$, $\mathbf{x} \mapsto f_{\mathbf{v}}(\mathbf{x})$ can be affine but can never be constant on any ball intersecting $\mathrm{supp}(\mu)$. In other words, $\nabla f_{\mathbf{v}}$ can be a constant function on U, but it is not identically zero on any component. We define $E_{\mathbf{v},\mathbf{0}}$ as in (4.2). We see that for all $\mathbf{v} \in \mathbb{S}^{d-1}$, $E_{\mathbf{v},\mathbf{0}}$ is a proper analytic subvariety with dimension $< k$. Theorem 3.6 shows that μ uniformly decays near the compact family $\{E_{\mathbf{v},\mathbf{0}} : \mathbf{v} \in \mathbb{S}^{d-1}\}$, in other words there exists some constants $\eta > 0$ such that

$$\mu(\{\mathbf{x} : |P_{\mathbf{v}}(\mathbf{x})| < \delta\}^{\delta}) \ll \delta^{\eta} \tag{4.10}$$

uniformly for $\delta > 0$ and $\mathbf{v} \in \mathbb{S}^{d-1}$.

By assumption, there exists $\sigma > 0$ such that $|\widehat{\mu}(\boldsymbol{\xi}')| \ll |\boldsymbol{\xi}'|^{-\sigma}$ for $\boldsymbol{\xi}' \in \mathbb{R}^k \setminus \{0\}$. Fix $\gamma \in (1,2)$. Since we are not concerned with optimising the exponent here, the precise choice of γ does not matter, so we can choose $\gamma = 1.5$ for concreteness. Fix $0 < \tau < 1$ small enough that $\gamma - 1 - \tau > 0$. It suffices to prove polynomial Fourier decay for μ_f at the geometric sequence of frequencies $\boldsymbol{\xi} = 2^{\gamma n}\mathbf{v}$, uniformly for $\mathbf{v} \in \mathbb{S}^{d-1}$, so we fix any such $\mathbf{v}$. For large $n \in \mathbb{N}$, we divide the set of dyadic cubes in $\mathcal{D}_n$ which intersect $\mathrm{supp}(\mu)$ into two sets, called $\mathcal{C}_n$ and $\mathcal{C}'_n$, where $\mathcal{C}_n$ consists of those which intersect the $2^{-\tau n}$-neighbourhood of $\{\mathbf{x} : |P_{\mathbf{v}}(\mathbf{x})| < 2^{-\tau n}\}$ and $\mathcal{C}'_n$ consists of those which do not. We may assume n is sufficiently large that all cubes in $\mathcal{C}_n$ are contained in the C-neighbourhood of $E_{\mathbf{v},\mathbf{0}}$, and that all cubes in $\mathcal{C}_n \cup \mathcal{C}'_n$ are contained in U, the domain of f. Let $\mu_{\omega,n}$ be associated with a cube of level n. Consider $\boldsymbol{\xi}_{\omega,n} \in \mathbb{R}^k$, $r_{\omega,n} \asymp 2^{-n}$ (the scaling of $|\mu_{\omega,n}|$) and $O_{\omega,n} \in O_k(\mathbb{R})$. Recalling the relationship between $\boldsymbol{\xi}_{\omega',n}$ and $P_{\mathbf{v}}$ from Claim 4.3 we see from (4.10) that the magnitude of $\boldsymbol{\xi}_{\omega',n}$ associated with a cube in $\mathcal{C}'_n$ is $\gg 2^{(\gamma-\tau)n}$.

Starting now from Lemma 4.1,

$$
\begin{aligned}
|\widehat{\mu_f}(\boldsymbol{\xi})| &\leqslant \sum_{D_n\in\mathcal{C}_n}\sum_{\mu_{\omega,n}\sim D_n}\left|\widehat{\mu}_{\omega,n}(\boldsymbol{\xi}_{\omega,n})\right| + \sum_{D'_n\in\mathcal{C}'_n}\sum_{\mu_{\omega',n}\sim D'_n}\left|\widehat{\mu}_{\omega',n}(\boldsymbol{\xi}_{\omega',n})\right| + O(|\boldsymbol{\xi}|/2^{2n}) \\
&\ll 2^{-\tau\eta n} + \sum_{D'_n\in\mathcal{C}'_n}\sum_{\mu_{\omega',n}\sim D'_n}|\mu_{\omega',n}||\widehat{\mu}(r_{\omega',n}O_{\omega',n}(\boldsymbol{\xi}_{\omega',n}))| + |\boldsymbol{\xi}|^{1-2/\gamma} \\
&\ll |\boldsymbol{\xi}|^{-\tau\eta/\gamma} + 2^{-(\gamma-1-\tau)\sigma n} + |\boldsymbol{\xi}|^{1-2/\gamma} \\
&\ll |\boldsymbol{\xi}|^{-\min\{\tau\eta/\gamma,(\gamma-1-\tau)\sigma/\gamma,2/\gamma-1\}},
\end{aligned}
$$

as required. □

4.4. **Lift to the graph.**

Proof of Theorem 3.3. Consider the map $T\colon \mathbf{x}\mapsto(\mathbf{x},f(\mathbf{x}))$. We consider the function $P^T_{\mathbf{v}}$ for some $\mathbf{v}\coloneqq(\mathbf{v}_k,\mathbf{v}_d)\in\mathbb{S}^{k+d-1}$ with $\mathbf{v}_k,\mathbf{v}_d$ the corresponding components in the first k and last d coordinates in $\mathbb{R}^{k+d}$ defined as

$$P^T_{\mathbf{v}}(\mathbf{x})\coloneqq \mathbf{v}_k + P_{\mathbf{v}_d}(\mathbf{x}) = \mathbf{v}_k + |\mathbf{v}_d|P_{\mathbf{v}_d/|\mathbf{v}_d|}(\mathbf{x}). \tag{4.11}$$

This is well-defined because $\mathbf{v}_d$ is not the zero vector by the hypothesis.

First assume (1). If $|P^T_{\mathbf{v}}|$ were a constant function of $\mathbf{x}$, then we see that the function defined by

$$\tilde{f}\colon \mathbf{x}\mapsto |\mathbf{v}_d|f(\mathbf{x}) + (\mathbf{v}_k,\mathbf{x})\mathbf{v}_d/|\mathbf{v}_d|$$

would be conical. This would mean that f is the sum of a conical function and an affine function, contradicting the hypothesis. So we have shown that $|P^T_{\mathbf{v}}|$ is not constant. The family $\{|P^T_{\mathbf{v}}(\cdot)|^2-r^2\}_{\mathbf{v},r}$ is still a compact family. However, it contains the zero function if we allow $\mathbf{v}_d=\mathbf{0}$. Nonetheless, letting $\varepsilon>0$, we see that after restricting to $|\mathbf{v}_d|>\varepsilon$, we have a compact family $\mathcal{F}_\varepsilon$. Therefore as in the proof of Theorem 3.1 (1), we have polynomial Fourier decay with uniform exponent and constant over directions $(\mathbf{v}_k,\mathbf{v}_d)\in\mathbb{S}^{k+d-1}$ with $|\mathbf{v}_d|\geqslant\varepsilon$. Let $0<\varepsilon<\varepsilon'$. The function $P_{(\varepsilon/\varepsilon')\mathbf{v}_d}$ corresponding to the function f is the same as the function $P_{\mathbf{v}_d}$ corresponding to the rescaled function $(\varepsilon'/\varepsilon)f$. Therefore the pushforward by the function $(\varepsilon'/\varepsilon)f$ in direction $(\mathbf{v}_k,\mathbf{v}_d)$ with $|\mathbf{v}_d|\geqslant\varepsilon$ has polynomial Fourier decay with the same constant and exponent as for the pushforward by the function f in direction $(\mathbf{v}_k,(\varepsilon/\varepsilon')\mathbf{v}_d)$ (noting that $|(\varepsilon/\varepsilon')\mathbf{v}_d|\geqslant\varepsilon'$). But scaling f by ε'/ε has the effect of scaling the implicit constant of Fourier decay for the pushforward by a factor depending on ε'/ε while keeping the exponent the same. This establishes the polynomial Fourier decay with a uniform decaying exponent (but possibly non-uniform constant), as in the statement of Theorem 3.3.

Next, we assume (2) instead of (1). Then the non-degeneracy of f implies that $P_{\mathbf{v}_d}$ is non-constant, which by the definition of $P_{\mathbf{v}}^T(\mathbf{x})$ from (4.11) tells us that $P_{\mathbf{v}}^T(\mathbf{x})$ is non-constant. Therefore if we restrict to $|\mathbf{v}_d| \geqslant \varepsilon$ then $\{P_{\mathbf{v}}^T(\cdot) - \mathbf{y}\}_{\mathbf{v},\mathbf{y}}$ forms a compact family without the zero function, so as above (and as in the proof of Theorem 3.1 (2)) we get the desired polynomial Fourier decay. □

Proof of Proposition 3.5. Let $\mathbf{v} := (\mathbf{v}_k, \mathbf{v}_d) \in \mathbb{S}^{k+d-1}$. If $\mathbf{v}_d = \mathbf{0}$ then $P_{\mathbf{v}}^T$ is constant and non-zero. Define $P_{\mathbf{v}}^T$ as in (4.11). If $\mathbf{v}_d \neq \mathbf{0}$ then $P_{\mathbf{v}}^T$ is non-constant by the non-degeneracy assumption. Therefore $\{(P_{\mathbf{v}}^T)^{-1}(\mathbf{0})\}_{\mathbf{v}\in\mathbb{S}^{k+d-1}}$ forms a compact family that does not contain the zero function. Thus as in the proof of Theorem 3.1 (3) (using the polynomial Fourier decay of μ), we get the desired polynomial Fourier decay for μ_T. □

5. Uniform measure Łojasiewicz inequality: proof of Theorem 3.6

We need the following lemma showing that affinely irreducible self-similar measures cannot 'see' proper analytic varieties.

Lemma 5.1. *Let μ be a self-similar measure supported inside a connected open set $U \subset \mathbb{R}^k$ and let $f_1, \ldots, f_n \colon U \to \mathbb{R}$ be real analytic functions. Consider the analytic subvariety*

$$M := \{\mathbf{x} \in U : f_1(\mathbf{x}) = \cdots = f_n(\mathbf{x}) = 0\}$$

and assume that $\mu(M) > 0$. Then there is a (possibly full-dimensional) affine subspace L such that $L \cap U \subseteq M$ and $\mu(L) = 1$.

Proof. If $M = U$ then we are done, so assume $M \neq U$. A classical result of Whitney [63, 64] (see also [41, Claim 2]) tells us that M is at most a countable union of proper analytic submanifolds. We can therefore assume without loss of generality that M is a proper analytic submanifold of dimension $< k$, without singularities.

We first assume for a contradiction that μ is affinely irreducible. Then for some small enough $\delta > 0$, all affine linear subspaces must miss at least one δ-branch of μ. The analyticity of M indicates that inside each δ-ball B, we have that $B \cap M$ is contained in the $C\delta$-neighbourhood of an affine subspace, where $C = o_\delta(1)$ uniformly across all points on M inside any fixed compact set. This compactness is not a restriction because the support of μ is compact. Therefore, we see that for some small enough $\delta > 0$, M must miss at least one δ-branch of μ. This implies that $\mu(M^\delta) < \rho$ for some $\rho < 1$, where M^δ denotes the δ-neighbourhood of M. For each δ-ball hosting δ-branches of μ, we can zoom it to have a unit length and see that M inside this ball will miss at least one δ^2-branch of μ. This holds inside all δ-branches of μ, so we conclude that

$$\mu(M^{\delta^2}) \leqslant \rho^2.$$

Iterating, we see that $\mu(M^r) \ll r^\eta$ for some $\eta > 0$ and $r \in (0,1)$. This contradicts the fact that $\mu(M) > 0$.

We have shown that μ must be affinely reducible and thus gives positive measure to some proper affine subspace. We can then find the smallest subspace L that contains the support of μ. If $L \cap U \not\subseteq M$, then $M \cap L$ is a proper analytic subvariety of L, and $\mu(M \cap L) = \mu(M) > 0$, so we can repeat the argument above to get a contradiction. Therefore we see that M must contain $L \cap U$. This finishes the proof. □

We now push the lemma one step further, showing that the same conclusion holds for Cartesian products of self-similar measures. Note that such product measures may not be self-similar.

Lemma 5.2. *Let μ be a Cartesian product of possibly different self-similar measures. Suppose that μ is supported inside a connected open set $U \subset \mathbb{R}^k$ and let $f_1, \dots, f_n \colon U \to \mathbb{R}$ be real analytic functions. Consider the analytic subvariety $M \coloneqq \{\mathbf{x} \in U : f_1(\mathbf{x}) = \cdots = f_n(\mathbf{x}) = 0\}$ and assume that $\mu(M) > 0$. Then there is a (possibly full-dimensional) affine subspace L such that $L \cap U \subset M$ and $\mu(L) = 1$.*

Proof. Let $\delta > 0$ be some small number. We can iterate the self-similar systems of the components of μ to obtain $\asymp \delta$-branches. Then μ is supported inside the Cartesian products of such $\asymp \delta$-branches, each of which has size $\asymp \delta$. We call them δ-product branches.

Now, we assume that μ is affinely irreducible. We claim that if δ is small enough, any hyperplane H has at least one such $\asymp \delta$-product branch outside of the δ-neighbourhood of H.

If not, then for all sufficiently small δ, all $\asymp \delta$-branches must intersect some hyperplane H_δ. Since the support of μ is compact, we can assume the Hausdorff convergence of such H_{δ_i} along some sequence $\delta_i \to 0$. Denote the limiting hyperplane to be H_*. This H_* has the property that for any $\delta > 0$, H_*^δ contains all the $\asymp \delta'$-product branches for some sufficiently small $\delta' > 0$. This implies that the support of μ must be contained in H_*.

The rest of the arguments are the same as those in Lemma 5.1, and we omit further repetitions. □

Next, we make the following simple observation that Cartesian product of affinely irreducible self-similar measures is also affinely irreducible.

Lemma 5.3. *Let μ be an affinely irreducible self-similar measure on $\mathbb{R}^k$. Then for any integer $l > 0$, the measure $\mu_l = \mu \times \cdots \times \mu$ (l-times Cartesian product) is also affinely irreducible.*

Proof. Suppose that the result does not hold for some $l \geqslant 1$. Consider the l-times Cartesian product μ_l. Suppose that this measure is affinely reducible, then there is a proper hyperplane $L \subset \mathbb{R}^{lk}$ so that μ_l is supported on L.

Let $1 \leqslant j \leqslant l$ be an integer. Let L_j the subspace $\{\mathbf{0}\} \times \cdots \times \mathbb{R}^k \times \cdots \times \{\mathbf{0}\} \subset \mathbb{R}^{lk}$ where the component R^k is on the j-th place. Then we have $\operatorname{supp}(\mu) = L_j \cap \operatorname{supp}(\mu_l) \subset L_j \cap L$. Since μ is affinely irreducible we see that $L_j \cap L = \mathbb{R}^k$. This holds for all j. We see that L must be $\mathbb{R}^{lk}$ which is not proper. This contraction implies that μ_l is affinely irreducible. □

Since $\mathcal{A}$ is a compact family, for any compact K, we consider Taylor expansions of $f \in \mathcal{A}$ at $x \in K$. It is a sum of monomials of integer degrees.

Lemma 5.4. *Suppose that $\mathcal{A}$ does not contain the zero function. Let $K \subset \mathbb{R}^k$ be compact. Then there is some $D > 0, c > 0$ such that at least one of the coefficients of f at $\mathbf{x} \in K$ with degree at most D has absolute value bigger than c.*

Proof. Write $\mathcal{A}$ as f_I with an index set I. Suppose that the result does not hold. Then there is a sequence $D_i \to \infty, c_i \to 0, I_i \in I, \mathbf{x}_i \in K$ such that the Taylor expansion of f_{I_i} at $\mathbf{x}_i$ has all its coefficients for monomials with degree at most D_i with absolute value at most c_i. As $\mathcal{A}$ is a compact family, we can assume the uniform convergence of $f_{I_i} \to f \in \mathcal{A}$ as well as $\mathbf{x}_i \to \mathbf{x} \in K$ as $i \to \infty$. We consider the Taylor expansion of f at $\mathbf{x}$. By the multivariable Cauchy integral formula, the Taylor coefficients of f is the limit of the corresponding coefficients of f_i, which is zero by assumption. We see that f expanded to be zero at $\mathbf{x}$ and this implies that f is zero. This contradicts the fact that $\mathcal{A}$ does not contain the zero function. □

We can then scale the family of functions in $\mathcal{A}$ such that $c = 1$. We will assume this for the rest of this section.

Let D be as in the above lemma. Let $H_{k,D}$ be the collection of monomials in k variables with degree at most D listed in some fixed order. Let there be L many such monomials. For each $x_1, \dots, x_k \in \mathbb{R}$, let

$$H_{k,D}(x_1, \dots, x_k) \in \mathbb{R}^L$$

be the evaluation of $H_{k,D}$ at $x_1, \dots, x_k$.

Let $\mathbf{x}_1, \dots, \mathbf{x}_L$ be real vectors in $\mathbb{R}^k$. We define $M = M(\mathbf{x}_1, \dots, \mathbf{x}_k)$ to be the real $L \times L$ matrix with row vectors

$$H_{k,D}(\mathbf{x}_1), \dots, H_{k,D}(\mathbf{x}_L).$$

We consider the function $h_{k,D} \colon \mathbb{R}^{Lk} \to R$ defined by

$$h_{k,D} \colon (\mathbf{x}_1, \dots, \mathbf{x}_L) \mapsto \det M(\mathbf{x}_1, \dots, \mathbf{x}_L).$$

We claim that $h_{k,D}$ is not a constant polynomial. Indeed, if it were a constant polynomial, it has to be the zero polynomial since inserting zeros in $h_{k,D}$ will obtain zero. Consider the map $H_{k,D} \colon \mathbb{R}^k \to \mathbb{R}^L$ which defines a k-manifold M_H in $\mathbb{R}^L$. If there are L many points (as vectors) on M_H that are linearly independent, then there exists a choice $\mathbf{x}_1, \dots, \mathbf{x}_L$ so that the matrix $M(\mathbf{x}_1, \dots, \mathbf{x}_L)$ is not singular (Given that monomials are linearly independent as functions, this should be clear. We will provide a simple argument below). Therefore $h_{k,D}$ cannot be constant.

We proceed to find such points. Let $y_1 \in M_H$ be a point. Then y_1 cannot be the origin because the constant 1 is included as the degree zero monomial. We can find $y_2 \in M_H$ so that y_1, y_2 are linearly independent because otherwise M_H must be contained in a homogeneous line which is not the case. Next, we can find linearly independent $y_1, y_2, y_3 \in M_H$ for otherwise M_H must be contained in a two-dimensional and homogeneous flat manifold which is not the case. Continue this argument. We see that there are linearly independent $y_1, \dots, y_L$ in M_H for otherwise M_H is contained in a proper homogeneous hyperplane. That is to say that monomials forming $H_{k,D}$ must be linearly dependent which is not the case.

We conclude that $V_{k,D} = \{h_{k,D} = 0\} \subset \mathbb{R}^{Lk}$ is a proper real algebraic variety. Using Lemmas 5.1, 5.2 and 5.3 we see that $\mu^L(V_{k,D}) = 0$ where μ^L is the L-fold Cartesian product of μ. Since $h_{k,D}$ is a polynomial, hence uniformly continuous on each compact $K \subset \mathbb{R}^{Lk}$, for $\varepsilon > 0$, there is some $\delta > 0$ such that

$$V^{\varepsilon}_{k,D} \cap K \supset \{|h_{k,D}| < \delta\} \cap K.$$

Let K be the support of μ^L in above. For some $\varepsilon > 0$, we see that at least one ε-branch of μ^L is disjoint from $V^{\varepsilon}_{k,D}$. Otherwise, we must have

$$\operatorname{supp}(\mu^L) \subset V_{k,D}$$

and this is impossible because $\mu^L(V_{k,D}) = 0$.

Notice that $\operatorname{supp}(\mu^L)$ is a Cartesian product of $\operatorname{supp}(\mu)$. Since μ is also self-similar, we see that an ε-branch of μ^L is the Cartesian product of L many ε-branches of μ. Thus, we see that there is $\delta > 0$ depending on ε and ε-branches $S_1, \ldots, S_L$ of μ so that as long as $\mathbf{x}_i \in S_i, i \in \{1, \ldots, L\}$,

$$(\mathbf{x}_1, \ldots, \mathbf{x}_L) \notin \{|h_{k,D}| < \delta\}.$$

This is saying that

$$|h_{k,D}(\mathbf{x}_1, \ldots, \mathbf{x}_L)| \geqslant \delta.$$

5.1. **Polynomials.** We first show the result for the case when $\mathcal{A}$ is a collection of polynomials with a uniformly bounded degree. The proof already features the key ideas.

Proposition 5.5. *Let $\mathcal{P}$ be a collection of polynomials in k variables with degree at most D. Suppose further that $\mathcal{P}$ is a compact collection viewing each polynomial as a vector composed of the polynomial's coefficients. Let μ be an affinely irreducible self-similar measure in $\mathbb{R}^k$. Then for all $\varepsilon > 0$, there are c, δ such that for all $P \in \mathcal{P}$,*

$$\mu(\{|P| < r\}^{r^\varepsilon}) \leqslant cr^\delta.$$

Proof. We can assume that, for any fixed compact set K, the Taylor expansion of $P \in \mathcal{P}$ at each $\mathbf{x} \in K$ has at least one coefficient bigger than one.

Let $0 < r < \rho < 1$. Consider a ρ-branch of μ. Let $B = B_\rho$ be a ball of radius $\asymp \rho$ that contains this ρ-branch. Without loss of generality, we take B to be centred at the origin which is fixed by the similarities of μ. Let $\eta > 0$ be a small number that will be specified later.

Suppose that there exists $P \in \mathcal{P}$ such that all $\eta\rho$-branches of μ in B intersect $\{|P| < r\}$ nontrivially. We write P as

$$P(T_1, \ldots, T_k) = \sum_I c_I T_I$$

where $c_I \in \mathbb{R}$ and T_I is a monomial with respect to $T_1, \ldots, T_k$ of degree $|I|$. For convenience, we rescale B to be the unit ball. Then $\eta\rho$-branches of μ in B become η-branches of μ after the rescaling. The polynomial P becomes

$$P_\rho(T_1, \ldots, T_k) \coloneqq P(T_1/\rho, \ldots, T_k/\rho) = \sum_I c_I \rho^{|I|} T_I.$$

By assumption, for each η-branch S of μ in the ball, there is some $\mathbf{x} \in S$ so that $|P_\rho(\mathbf{x})| < r$. We can choose such $\mathbf{x}_1, \ldots, \mathbf{x}_L$ (one in each η-branch) such that $|P_\rho(\mathbf{x}_j)| < r$ for each $\mathbf{x}_j$. To do this, we must be sure that there are enough ($\geqslant L$) η-branches. Since L is fixed, this can be guaranteed by choosing η to be a small enough number (depending only on k, D, μ).

We then see that

$$M(\mathbf{x}_1, \ldots, \mathbf{x}_L)\mathbf{c} = \mathbf{y}$$

where $\mathbf{c}$ is the real column vector of length L consisting of the coefficients $\rho^{|I|}c_I$ of P_ρ and $\mathbf{y}$ is a column vector of length L with $\|\mathbf{y}\|_\infty < r$. Assuming M is invertible, we then see that

$$\mathbf{c} = M^{-1}\mathbf{y}.$$

By Cramer's rule, we have

$$M^{-1} = \frac{\mathrm{adj}M}{h_{k,D}(\mathbf{x}_1, \ldots, \mathbf{x}_L)}.$$

If η is small enough (again depending only on k, D, μ), there is some $\eta' > 0$ (depending on the same parameters) so that we could have chosen the η-branches $S_1, \ldots, S_L$ of μ such that for all $\mathbf{x}_j \in S_j$,

$$|h_{k,D}(\mathbf{x}_1, \ldots, \mathbf{x}_L)| \geqslant \eta'.$$

In particular, $M(\mathbf{x}_1, \ldots, \mathbf{x}_L)$ is invertible. On the other hand, the matrix $\mathrm{adj}M$ has absolutely bounded entries. Thus all entries of M^{-1} are $O(1)$ for $\mathbf{x}_j \in S_j$. As long as η is fixed, this $O(1)$ depends only on k, D, μ and not on the choice of $\mathcal{P}$. We see that all entries of $\mathbf{c}$ are $O(r)$. For any one such entry, we have $c_I\rho^{|I|} = O(r)$ and this implies that

$$c_I = O(r/\rho^{|I|}).$$

Since at least one c_I is larger than one, we see that there is a constant $c' > 0$ so that for all $r > 0$ and $\rho > c'r^{1/D}$, inside any ρ-branch of μ, for each $P \in \mathcal{P}$, at least one $\eta\rho$-branch of μ is disjoint with $\{|P| < r\}$.

Fix one P and fix $r > 0$. We can apply the above conclusion by first considering η-branches of μ and for each η-branch intersecting $\{|P| < r\}$, we consider η^2-branches inside this η-branch. We repeat this procedure t times for some t with $c'r^{1/D} < \eta^t < c'\eta^{-1}r^{1/D}$. Thus $t \gg |\log r|$. Consider the η-branches of μ and let p be the smallest probability weight of such branches. We then see that

$$\mu(\{|P| < r\}^{\eta^t}) \ll (1-p)^t \ll r^{\eta''}. \tag{5.1}$$

for some $\eta'' > 0$ and uniformly for all $P \in \mathcal{P}$, $r > 0$. After matching the parameters, this finishes the proof of the result for some specific ε_0. Then the result holds automatically for all $\varepsilon > \varepsilon_0$. To see the result for each $0 < \varepsilon < \varepsilon_0$, observe that in (5.1), we can choose possibly smaller values for t, with $t \gg |\log r|$[5] and the implicit multiplicative constant can be arbitrarily small. □

[5]For example, $t = \gamma|\log r|$ with small γ.

5.2. **Analytic functions.** We now prove the extension of the above result to real analytic functions, i.e. we prove Theorem 3.6.

We can repeat the proof of the polynomial case. Now we write each $f \in \mathcal{A}$ as

$$f(T_1, \dots, T_k) = \sum_{|I| \leqslant D} c_I T_I + \sum_{|I| > D} c_I T_I$$

We can then write

$$f_\rho(T_1, \dots, T_k) = \sum_{|I| \leqslant D} c_I \rho^{|I|} T_I + \sum_{|I| > D} c_I \rho^{|I|} T_I$$

Because $\mathcal{A}$ is a compact family, the higher order part is bounded uniformly by $O(\rho^{D+1})$ (due to the Cauchy integral formula). The bound is allowed to depend on D. As D is some fixed number, this dependence does not show any significance. From here, the argument is similar to that of the polynomial case.

As in the polynomial case, if in some ρ-branch of μ, for some $f \in \mathcal{A}$, all its $\eta\rho$-branches intersect $\{|f| < r\}$ nontrivially, we obtain

$$M(\mathbf{x}_1, \dots, \mathbf{x}_L)\mathbf{c} = \mathbf{y} + O(\rho^{D+1})$$

where $\mathbf{c}$ is the column real vector consists of the coefficients $\rho^{|I|} c_I$ and $\|\mathbf{y}\|_\infty < r$. The extra error $O(\rho^{D+1})$ comes from the tail sum of f_ρ. Then as in the polynomial case, we deduce that for I with $|I| \leqslant D$ and $\rho \gg r^{1/(D+1)}$,

$$c_I = O(\rho^{D+1}/\rho^D) = O(\rho).$$

We need $\rho \gg r^{1/(D+1)}$ to make sure that

$$\rho^{D+1} \gg r$$

so that $\mathbf{y} + O(\rho^{D+1})$ is $O(\rho^{D+1})$. Knowing that some c_I is $\gg 1$, we conclude the result in the same way as in the polynomial case. From here, Theorem 3.6 follows.

6. Fourier decay outside sparse frequencies: proof of Theorem 3.9

Recall the definition of Fourier decay outside sparse frequencies from Definition 3.8. Since the Fourier transform of μ is Lipschitz, this property is clearly equivalent to saying that $\{\boldsymbol{\xi} \in \mathbb{R}^k : |\widehat{\mu}(\boldsymbol{\xi})| \geqslant R^{-\delta}, |\boldsymbol{\xi}| < R\}$ can be covered by $\ll R^\varepsilon$ balls in $\mathbb{R}^k$ of radius 1. For the proof of Theorem 3.9 we need to introduce some more notation. Given $\mathbf{x} \in \mathbb{R}^k$ and $r > 0$, the open ball of radius $r > 0$ centred at $\mathbf{x}$ is denoted

$B(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^k : |\mathbf{y} - \mathbf{x}| < r\}$. For a set $W \subset \mathbb{R}^k$ and $r > 0$ we denote the r-neighbourhood by

$$W^{(r)} := \{y \in \mathbb{R}^k : |x - y| < r \text{ for some } x \in W\}.$$

We make the following definition, giving a condition which is weaker than Khalil's uniform affine non-concentration condition [33, (1.3)] (which corresponds to the $c = 1$ case below).

Definition 6.1. *We say a Borel probability measure μ on $\mathbb{R}^k$ is* inner-affinely non-concentrated *if there exist $c \geqslant 1$ and $\phi\colon (0,1) \to \mathbb{R}$ with $\phi(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that for all $\mathbf{x} \in \operatorname{supp}(\mu)$, $0 < r < 1$, $\varepsilon \in (0,1)$ and all proper affine hyperplanes $W \subset \mathbb{R}^k$,*

$$\mu(W^{(\varepsilon r)} \cap B(\mathbf{x}, r)) \leqslant \phi(\varepsilon)\mu(B(\mathbf{x}, cr)), \tag{6.1}$$

where $W^{(\varepsilon r)}$ is the εr-neighbourhood of W.

We prove the following result about self-similar measures, which in particular shows that the conclusions of Theorems 6.6, 6.8 and 6.4 below hold when μ is an affinely irreducible self-similar measure. The proof uses a somewhat similar strategy to the proof of [26, Proposition 2.2], and Lemma 5.1.

Theorem 6.2. *Let μ be an affinely irreducible self-similar measure on $\mathbb{R}^k$. Then there exists $\alpha > 0$ such that for all $c > 1$, $\mathbf{x} \in \operatorname{supp}(\mu)$, $0 < r < 1$, $\varepsilon \in (0,1)$ and all proper affine hyperplanes $W \subset \mathbb{R}^k$,*

$$\mu(W^{(\varepsilon r)} \cap B(\mathbf{x}, r)) \ll_{k,\mu,c} \varepsilon^\alpha \mu(B(\mathbf{x}, cr)).$$

In particular, μ is inner-affinely non-concentrated.

The following basic fact will be used in the proof of Theorem 6.2.

Lemma 6.3. *Let μ be a finite compactly supported Borel measure on $\mathbb{R}^k$ whose support does not lie in any proper affine subspace. Then there exists $\delta, \gamma \in (0,1)$ such that for all proper affine subspaces K of $\mathbb{R}^k$,*

$$\mu(K^{(\gamma)}) \leqslant (1 - \delta)\mu(\mathbb{R}^k).$$

Proof. Assume for contradiction this is false. Then there exists a sequence $(S_n)_{n=1}^\infty$ of $k-1$ dimensional affine subspace of $\mathbb{R}^k$ such that $\mu(S_n^{(1/n)}) \geqslant \mu(\mathbb{R}^k) - 1/n$. Let B be a ball containing the support of μ. By compactness we can find a subsequence of S_n which converges to some $k-1$ dimensional affine subspace S, and take a further subsequence (call it $(S_{k_n})_{n=1}^\infty$) such that $S_{k_n}^{(1/k_n)} \cap B \supseteq S_{k_{n+1}}^{(1/k_{n+1})} \cap B$ for all $n \in \mathbb{N}$. Therefore since $\mu(S_{k_n}^{(1/k_n)}) \geqslant \mu(\mathbb{R}^k) - 1/k_n$ for all n, we must have $\mu(S_{k_n}^{(1/k_n)}) = \mu(\mathbb{R}^k)$ for all n. But this means that $\mu(S) = \mu(\mathbb{R}^k)$, contradicting our assumption that μ is not supported in a proper affine subspace. □

Proof of Theorem 6.2. Without loss of generality we may assume that the support of μ is contained in a ball of diameter 1. Let δ, γ be as in Lemma 6.3 for μ. Fix $c > 1$. Fix an IFS defining μ and let $r_{\min} \in (0,1)$ be its smallest contraction ratio. Write $\mathcal{F}$ for the set of all proper affine hyperplanes in $\mathbb{R}^k$. Fix $K \in \mathcal{F}$, $\mathbf{x} \in \operatorname{supp}(\mu)$ and $0 < r < 1$.

Claim: For all $n \geqslant 1$, if we let $r_n \coloneqq (c-1)r(\gamma r_{\min}/4)^{n-1}$ then

$$\mu(K^{(r_n)} \cap B(\mathbf{x}, r + r_n)) \leqslant (1-\delta)^{n-1}\mu(B(\mathbf{x}, cr)).$$

Proof of claim: The proof goes by induction. The $n = 1$ case holds since $K^{(r_1)} \cap B(\mathbf{x}, r + r_1) \subseteq \mu(B(\mathbf{x}, cr))$, which is true because $(c-1)r \geqslant r_1 \geqslant r_2 \geqslant \cdots$. Assume that the claim holds for some $n \geqslant 1$. Write $\mu = \sum_{w\in\Omega} p_w \mu_{f_w}$ where Ω is a subset of finite words from the alphabet of the IFS such that contraction ratios of the corresponding compositions f_w all lie in the interval $[r_n r_{\min}/2, r_n/2]$. Let

$$\Omega' \coloneqq \{w \in \Omega : \operatorname{supp}(\mu_{f_w}) \cap K^{(r_{n+1})} \cap B(\mathbf{x}, r + r_{n+1}) \neq \varnothing\}.$$

By our choice of r_n, if $w \in \Omega'$ then $\operatorname{supp}(\mu_{f_w}) \subseteq K^{(r_n)} \cap B(\mathbf{x}, r + r_n)$. Now by Lemma 6.3,

$$\begin{aligned}
\mu(K^{(r_{n+1})} \cap B(\mathbf{x}, r + r_{n+1})) &= \sum_{w\in\Omega'} p_w \mu(f_w^{-1}((K^{(r_{n+1})} \cap B(\mathbf{x}, r + r_{n+1}))) \\
&\leqslant \sum_{w\in\Omega'} p_w(1-\delta) \\
&\leqslant (1-\delta)\mu(K^{(r_n)} \cap B(\mathbf{x}, r + r_n)) \\
&\leqslant (1-\delta)^n \mu(B(\mathbf{x}, cr)),
\end{aligned}$$

where the last inequality was by the induction hypothesis. This completes the proof of the claim.

It follows from this claim that

$$\mu(K^{(r_n)} \cap B(\mathbf{x}, r)) \leqslant \mu(K^{(r_n)} \cap B(\mathbf{x}, r + r_n)) \leqslant (1-\delta)^n B(\mathbf{x}, cr).$$

We can increase the implicit multiplicative constant (in a way that depends on k, μ, c) so that the desired non-concentration property holds along a geometric sequence of scales with the value of α depending only on $\delta, \gamma, r_{\min}$, which in turn depend only on k and μ. By increasing the implicit constant further, the property holds at all scales. □

Khalil's proof of decay outside sparse frequencies under different non-concentration assumptions also goes through under the inner-affine non-concentration assumption, giving the following statement.

Theorem 6.4. *[follows from methods from [33]] Every inner-affine non-concentrated Borel probability measure on $\mathbb{R}^k$ has Fourier decay outside sparse frequencies.*

We now have all the pieces required for Theorem 3.9.

Proof of Theorem 3.9. It is straightforward to see that if a Borel probability measure is supported on a proper affine subspace of $\mathbb{R}^k$ then it cannot have Fourier decay outside sparse frequencies. The non-trivial direction is the converse implication, which follows from Theorem 6.4 and Theorem 6.2. $\square$

The remainder of this section is devoted to giving a bit more justification for how Khalil's methods give Theorem 6.4. One needs to work with discretised approximations of inner-affinely non-concentrated measures. For $j \geqslant 0$, recall that $\mathcal{D}_j$ is the natural dyadic decomposition of $\mathbb{R}^k$ into cubes of sidelength 2^{-j}. Given a Borel probability measure ν on $\mathbb{R}^k$, let

$$\nu_j \coloneqq \sum_{\lambda \in 2^{-j}\mathbb{Z}^j} \nu(\mathcal{D}_j(\lambda))\delta_\lambda$$

denote the scale-j discretisation of ν. Given a probability measure μ with finite support inside $2^{-j}\mathbb{Z}^k \coloneqq \{2^{-j}\mathbf{x} : \mathbf{x} \in \mathbb{Z}^k\}$, write

$$||\mu||_2 \coloneqq \sqrt{\sum_{\lambda \in 2^{-j}\mathbb{Z}^k} (\mu(\{\lambda\}))^2}.$$

We write μ^{*n} for the n-fold convolution of μ. They have the following property.

Lemma 6.5. *Let μ be an inner-affinely non-concentrated Borel probability measure with function ϕ, supported inside the open unit ball centred at $\mathbf{0} \in \mathbb{R}^k$. Then there exists $c > 1$ such that for all sufficiently large natural numbers r, all $\varepsilon \geqslant 2^{-r}$, all integers $j \geqslant 2$, all $\mathbf{w} \in \operatorname{supp}(\mu_{jr})$, all proper affine hyperplanes $W \subset \mathbb{R}^k$, and all $l \in \{0, 1, \ldots, j-2\}$,*

$$\mu_{jr}(W^{(\varepsilon 2^{-lr})} \cap \mathcal{D}_{lr}(\mathbf{w})) \ll \phi(2\varepsilon/c')\mu(B(\mathbf{w}, c2^{-lr})).$$

Here, c and the implicit constant depend only on k and the non-concentration parameters of μ.

Proof. Fix $c' > 1$ large enough that for all $j \in \mathbb{N}$, a dyadic cube of sidelength 2^{-j} has diameter strictly less than $c'2^{-j}$. Fix $r \in \mathbb{N}$ large enough that $2^r > c'$ and large enough that for all $j \in \mathbb{N}$ we have $\operatorname{supp}(\mu_{jr}) \subset 2^{-jr}\mathbb{Z}^k \cap B(\mathbf{0}, 1)$. Write $\rho_l \coloneqq 2^{-lr}$ for $l \geqslant 0$. For the rest of the proof we fix $\varepsilon \geqslant 2^{-r}$, $j \geqslant 2$, $\mathbf{w} \in \mathrm{supp}(\mu_{jr})$, $\mathbf{x} \in \mathrm{supp}(\mu) \cap \mathcal{D}_{jr}(\mathbf{w})$, $l \in \{0, 1, \ldots, j-2\}$, and a proper affine hyperplane $W \subset \mathbb{R}^k$.

Now suppose $\mathbf{v} \in 2^{-jr}\mathbb{Z}^k \cap W^{(\varepsilon\rho_l)} \cap \mathcal{D}_{lr}(\mathbf{w})$. Then by the choice of c', the diameter of $\mathcal{D}_{jr}(\mathbf{v})$ is less than $c'\rho_j$. Since $2^r > c'$ and $\varepsilon \geqslant 2^{-r}$ and $l \leqslant j-2$, we have $\mathcal{D}_{jr}(\mathbf{v}) \subset W^{(2\varepsilon\rho_l)}$. Similarly, $\mathcal{D}_{jr}(\mathbf{v}) \subseteq \mathcal{D}_{lr}(\mathbf{w}) \subset B(\mathbf{x}, c'\rho_l)$. Therefore

$$\mathcal{D}_{jr}(\mathbf{v}) \subset W^{(2\varepsilon\rho_l)} \cap B(\mathbf{x}, c'\rho_l) \subset B(\mathbf{w}, 2c'\rho_l).$$

Let the inner-affine non-concentration of μ hold with parameters $\phi(\varepsilon)$ and $c'' \geqslant 1$. Then

$$\begin{aligned}
\mu_{jr}(W^{(\varepsilon\rho_l)} \cap \mathcal{D}_{lr}(\mathbf{w})) &= \sum_{\mathbf{v} \in 2^{-jr}\mathbb{Z}^k \cap W^{(\varepsilon\rho_l)} \cap \mathcal{D}_{lr}(\mathbf{w})} \mu(\mathcal{D}_{jr}(\mathbf{v})) \\
&\leqslant \mu(W^{(2\varepsilon\rho_l)} \cap B(\mathbf{x}, c'\rho_l)) \\
&\ll \phi(2\varepsilon/c')\mu(B(\mathbf{x}, c'c''\rho_l)) \\
&\leqslant \phi(2\varepsilon/c')\mu(B(\mathbf{w}, 2c'c''\rho_l)).
\end{aligned}$$

Taking $c := 2c'c''$ completes the proof. □

The following smoothing result for L^2 norms of convolutions of discretised measures is the key ingredient in the proof of Theorem 3.9. It says that convolving an arbitrary measure ν with an inner non-concentrated measure causes ν to spread out in the sense that, unless $||\nu||_2$ is already close to 0, there is a quantitative reduction in $||\nu||_2$ (this is called 'L^2-flattening'). Theorem 6.6 is very similar to Khalil's [33, Proposition 11.14].

Theorem 6.6. *For all $0 < \gamma < 1$ there exists $\tau > 0$ and $j_1, r \in \mathbb{N}$ such that for all integers $j \geqslant j_1$ the following holds. Let ν be an arbitrary probability measure supported inside $2^{-jr}\mathbb{Z}^k \cap B(0, 2^n)$ for some $n \in \mathbb{N}$ and let μ be an inner-affinely non-concentrated Borel probability measure supported inside the ball $B(0, 2^m)$ for some integer $m \geqslant 0$. If ν satisfies*

$$||\nu||_2^2 > 2^{-(1-\gamma)kjr+2kn}$$

then

$$||\mu_{jr} * \nu||_2 \leqslant 2^{km-\tau jr}||\nu||_2.$$

The proof of Theorem 6.6 uses tools from additive combinatorics such as the asymmetric Balog–Szemerédi–Gowers lemma due to Tao and Vu [61, Theorem 11.6], and Hochman's inverse entropy theorem for convolutions of measures on $\mathbb{R}^d$ [27]. However, the proof is very similar to the proof of [33, Proposition 11.14], in fact in our setting the proof is simpler because (unlike in [33, Definition 11.1]) the non-concentration hypothesis on our measure is uniform. Essentially the only place where the non-concentration hypothesis is used in the proof of [33, Proposition 11.14] is in the proof of [33, Lemma 11.8], which says that if a measure is affinely non-concentrated then its discretisation satisfies (a weaker version of) the condition in Lemma 6.5. In other words, while the assumptions on the original measures in Lemma 6.5 and [33, Lemma 11.8] are not comparable, the conclusion for the discretised measures in Lemma 6.5 is strictly stronger than the corresponding conclusion for discretised measures in [33, Lemma 11.8]. The only other place in the proof of [33, Proposition 11.14] where affine non-concentration of μ itself is used is in the proof of [33, Lemma 11.22]. However inner-affine non-concentration suffices here because we can simply ensure that the constant c in the statement of Lemma 6.5 is larger than the optimal constant. Thus, instead of copying many pages of proof with slight simplifications, we omit the details. Naturally, the constants in the conclusions of Theorem 6.8, Theorem 6.6, Lemma 6.5 and indeed Theorem 3.9 depend on the constants in Definition 6.1 satisfied by the measures in question.

Remark 6.7. *After sharing a draft version of this paper with A. Rutar, he observed that an L^2 smoothing result for inner-affinely non-concentrated measures which is very similar to Theorem 6.6 (and which in particular is strong enough to recover Theorem 3.9 when combined with our Theorem 6.2) can alternatively be derived from an inverse theorem for L^q norms of convolutions due to Shmerkin [54, Theorem 1.2]. Shmerkin's theorem and Khalil's theorem are closely related and appeared independently around the same time and with a similar proof. A different range of applications of Shmerkin's inverse theorem to self-similar measures were recently proved in [17].*

The next result, Theorem 6.8, says that repeatedly convolving a (discretised) inner-affinely non-concentrated measure with itself results in smoothening in the sense that the $||\cdot||_2$-norm of the level-j discretisation of the convolved measure decreases rapidly with j. It is proved by following very similar methods as used in [33, Theorem 11.3], but under assumptions that apply to affinely irreducible self-similar measures.

Theorem 6.8. *For every $\eta > 0$ there exist natural numbers n and j_0 such that the following holds for every inner-affinely non-concentrated Borel probability measure μ with compact support inside $\mathbb{R}^k$. For all integers $m \geqslant 0$ large enough that μ is supported inside the ball of radius*

2^m around the origin in $\mathbb{R}^k$, and every $j \geqslant j_0$,

$$||\mu_j^{*n}||_2^2 \ll 2^{2km(n-1)-(k-\eta)j}$$

with implicit constant depending only on k and the non-concentration parameters of μ. In particular, for all $P \in \mathcal{D}_j$,

$$\mu_j^{*(n+1)}(P) \ll 2^{kmn-(k-\varepsilon)j/2}.$$

Proof. The first statement follows from the quantitative estimate Theorem 6.6 by the short argument from [33, Section 11.8]. The second follows from the first by the proof of [43, Lemma 5.2]. □

Theorem 6.8 shows that various notions of dimension (which we will not define) of iterated convolutions μ^{*n} of an affinely irreducible self-similar measure μ on $\mathbb{R}^k$ tend to k as $n \to \infty$. In particular, this is the case for Frostman dimension, (lower) Hausdorff dimension, (lower) correlation dimension, and (lower) L^q dimension

Proof of Theorem 6.4. This follows from Theorem 6.8 and Chebyshev's inequality by the argument in [33, Section 11.9]. □

7. Self-conformal measures

7.1. **Self-conformal measures in the plane.**

Recall the definition of self-conformal measures from Section 2.6. For now we work in the (complex) plane. For brevity we introduce the following terminology.

Definition 7.1. *Let $\{\varphi_i : U \to U\}$ be a conformal IFS and ν an associated self-conformal measure.*

- *We say Φ and ν are* nonlinear *if there exists i such that φ_i is non-affine.*
- *We say Φ and ν are* curve-reducible *if there exists a complex neighbourhood V of $(0,1)$ and a finite family of analytic functions $f_j : V \to \mathbb{C}$ with non-vanishing derivative such that* $\mathrm{supp}(\nu) \subseteq \cup_j f_j((0,1))$[6]*, and* curve-irreducible *otherwise.*

A key goal of this section is the following result, which will follow by combining Theorem 7.6 below with a recent result of Algom, Rodriguez Hertz and Wang [5].

Theorem (Restatement of Theorem 1.5)**.** *Let ν be a nonlinear and curve-irreducible self-conformal measure, and assume the domain $D \subset \mathbb{C}$ of the contractions is a closed ball. Then ν has polynomial Fourier decay.*

[6]We call such a set $f_j((0,1))$ an *analytic curve*.

Self-similar measures like the Cantor–Lebesgue measure (or products of this measure with itself) may not be Rajchman; this is the reason for the nonlinearity assumption. Example 7.7 below will show why the curve-irreducibility assumption cannot be removed. Note that if $\dim_{\mathrm{H}}\mathrm{supp}(\nu) > 1$ then ν is curve-irreducible.

The theory of nonlinear self-conformal measures breaks into two cases according to the following definition. We say that a conformal IFS $\Phi = \{\varphi_i\}_{i\in I}$ on $\mathbb{C}$ is *conjugate to linear* if there exists a self-similar IFS $\{\psi_i\}_{i\in I}$ and a holomorphic diffeomorphism f from an open neighbourhood of the self-similar set to an open neighbourhood of the self-similar set so that $\varphi_i = f \circ \psi_i \circ f^{-1}$ for all $i \in I$. We will see that the tools in this paper prove very fruitful in the conjugate-to-linear case, and address point 2 below the statement of Theorem 1.1 in [5].

Lemma 7.2. *Let ν be a self-conformal measure with weights p_i for an IFS $\{\varphi_i\}$ which is conjugate to a self-similar IFS $\{\psi_i\}$ via a holomorphic diffeomorphism f, using notation as above.*

1. *If μ is the self-similar measure for $\{\psi\}$ with the same weights p_i, then $\nu = \mu_f$ (in this case we say ν and μ are conjugate).*
2. *If ν is a nonlinear self-conformal measure, then f is non-affine on every connected component of its domain which intersects* $\mathrm{supp}(\mu)$.

Proof. (1): For all Borel $A \subseteq \mathbb{R}^k$,

$$\mu_f(A) = \mu(f^{-1}A) = \sum_i p_i\mu(\psi_i^{-1} \circ f^{-1}(A)) = \sum_i p_i\mu(f^{-1} \circ \varphi_i^{-1}(A)) = \sum_i p_i(\mu_f)_{\varphi_i}(A). \tag{7.1}$$

By the definition of a self-conformal measure this shows that $\nu = \mu_f$.

(2): We denote the non-affine map by φ_1. Since ν is compactly supported there exists $\delta > 0$ such that the range of f contains the δ-neighbourhood of $\mathrm{supp}(\nu)$. Fix any $z \in \mathrm{supp}(\nu)$, so for some sequence $(i_n)_{n=1}^{\infty}$ we can write $\{z\} = \bigcap_{n=1}^{\infty} \varphi_{i_1} \circ \cdots \circ \varphi_{i_n}(\mathrm{supp}(\nu))$. Fix n large enough that the diameter of $\varphi_{i_1}\circ\cdots\circ\varphi_{i_n}(\mathrm{supp}(\nu))$ is less than δ. By the nonlinearity assumption on φ_1, either $\varphi_{i_1}\circ\cdots\circ\varphi_{i_n}$ or $\varphi_{i_1}\circ\cdots\circ\varphi_{i_n}\circ\varphi_1$ is non-affine on $B(z,\delta)$. In other words, either $f \circ \psi_{i_1} \circ \cdots \circ \psi_{i_n} \circ f^{-1}$ or $f \circ \psi_{i_1} \circ \cdots \circ \psi_{i_n} \circ \psi_1 \circ f^{-1}$ is non-affine on $f^{-1}(B(z,\delta))$. Since the ψ_i are affine, f is non-affine on $f^{-1}(B(z,\delta))$. Since $z \in \mathrm{supp}(\nu)$ was arbitrary and f is holomorphic, this completes the proof. □

We will use the following fact from complex analysis.

Lemma 7.3. *Let $U \subset \mathbb{C}$ be non-empty, open and connected and $f\colon U \to \mathbb{C}$ holomorphic. Then f is degenerate if and only if f is affine.*

Proof. The backward implication is trivial, so we prove the forward implication. Assume f is degenerate. This means that there is some non-trivial affine form L so that $L(x, y, u(x, y), v(x, y)) = 0$ constantly. Here u, v are the real and imaginary parts of f. We write out the affine form with real coefficients as

$$c_0 + c_x x + c_y y + c_u u + c_v v = 0.$$

We can make partial derivatives against x, y and see that

$$\begin{bmatrix} c_u & -c_v \\ c_v & c_u \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} -c_x \\ -c_y \end{bmatrix},$$

where we have used the Cauchy–Riemann equations ($u_x = v_y$, $u_y = -v_x$) for holomorphic functions to deal with partial derivatives of u, v. The determinant of the matrix cannot be zero unless $c_u = c_v = 0$; in this case, L indicates an affine relation between x, y which cannot exist. Therefore we can solve the above linear equations and see that u_x, u_y are constant functions. This implies that u is affine. Similarly, v is affine. Thus f is affine, as required. □

Lemma 7.3 allows us to prove the following Kaufman-type result for holomorphic pushforwards as a consequence of Theorem 3.1 (2).

Theorem 7.4. *Let μ be an affinely irreducible self-similar measure on $\mathbb{C}$, let $U \subset \mathbb{C}$ be a connected open neighbourhood of $\operatorname{supp}(\mu)$, and let $f\colon U \to \mathbb{C}$ be holomorphic and non-affine. Then μ_f has polynomial Fourier decay.*

Proof. By Lemma 7.3, f is non-degenerate, so μ_f has polynomial Fourier decay by Theorem 3.1 (2). □

Remark 7.5. *In [11, Theorem 1.6], we proved quantitative polynomial Fourier decay for nonlinear holomorphic pushforwards of self-similar measures in $\mathbb{C}$ with dimension greater than 1. Theorem 7.4, on the other hand, does not quantify the exponent, but works even for pushforwards of self-similar measures with small dimension.*

We are ready to apply Theorem 7.4 to prove our main contribution of this section.

Theorem 7.6. *Every curve-irreducible, nonlinear, and conjugate-to-linear self-conformal measure in $\mathbb{C}$ has polynomial Fourier decay.*

Proof. By Lemma 7.2 (1) we can write $\nu = \mu_f$ for a self-similar measure μ on $\mathbb{C}$ and a holomorphic map f. By Lemma 7.2 (2), f is non-affine on every connected component of its domain which intersects $\mathrm{supp}(\mu)$. Since ν is curve-irreducible, μ must be affinely irreducible. Since $\mathrm{supp}(\mu)$ is compact, fix $N \in \mathbb{N}$ large enough that the support of f contains the $r_{\max}^N$-neighbourhood of $\mathrm{supp}(\mu)$, where $r_{\max}$ is the largest contraction ratio of a map in the IFS $\{\psi_i\}$ defining μ. For all $\xi \in \mathbb{C}$, $|\widehat{\nu}(\xi)|$ can be bounded above by the sum of the magnitudes of the Fourier transform of $(\#\Phi)^N$ nondegenerate analytic images of μ at the frequency ξ. But $(\#\Phi)^N$ is a fixed constant independent of ξ, so Theorem 7.4 gives that ν has polynomial Fourier decay. □

We now have all the pieces to combine with [5] and give Theorem 1.5.

Proof of Theorem 1.5. If ν is conjugate to linear then the result follows from Theorem 7.6. If ν is not conjugate to linear, then (as noted in [5]) an application of the Poincaré–Siegel theorem [30, Theorem 2.8.2] similar to the proof of [4, Claim 6.1] gives that ν is not conjugate to linear via any C^2 diffeomorphism, and ν has polynomial Fourier decay by [5, Theorem 1.1].[7] □

For self-conformal measures supported inside analytic curves, one needs to be more careful. If the self-conformal measure is supported inside a straight line, then there is no Fourier decay along directions orthogonal to that line (irrespective of any nonlinearity conditions). More interestingly, we have the following example. Note that $z \mapsto z + i(z+1)^2$ is a non-degenerate function $\mathbb{C} \to \mathbb{C}$ (i.e. $\mathbb{R}^2 \to \mathbb{R}^2$), but that its restriction to the real line (which is the map from Remark 2.3 (4)) is degenerate when thought of as a function $\mathbb{R} \to \mathbb{R}^2$.

Example 7.7. *Let μ be the Cantor–Lebesgue measure on $[0,1] \subset \mathbb{C}$, i.e. the $(1/2, 1/2)$ self-similar measure for $\{\varphi_1, \varphi_2\}$ where $\varphi_1(z) = z/3$ and $\varphi_2(z) = (z+2)/3$. Fix a complex neighbourhood U of $[0,1]$ that is small enough that $f \colon U \to f(U)$ is a conformal diffeomorphism, where f is the quadratic polynomial $f(z) = z + i(z+1)^2$, noting that f lifts $[0,1]$ to a parabola. We can moreover choose U in such a way that $2 < |f'(z)| < 5$ for all $z \in U$ and that φ_1 and φ_2 map U into itself. The IFS*

$$\{f \circ \varphi_1 \circ f^{-1}, f \circ \varphi_2 \circ f^{-1}\}$$

on $f(U)$ can be seen to be uniformly contracting by a simple application of the chain rule. By the calculation from (7.1), μ_f is the self-conformal

[7] This result assumes that the set D in the definition of the self-conformal measure from Section 2.6 is a closed disc, but it is likely that this assumption can be relaxed, as described in the text after the statement of the theorem.

measure for this IFS and $(1/2, 1/2)$ *weights.*[8] *Moreover,* $\widehat{\mu_f}(3^n) = \widehat{\mu}(3^n)$ *takes a constant nonzero value independent of* $n \in \mathbb{N}$*, so* μ_f *is not Rajchman.*

More generally, if μ is a self-similar measure supported on $[0, 1]$ without polynomial Fourier decay, U is a complex neighbourhood of $[0, 1]$, and $f : U \to \mathbb{C}$ is a holomorphic map which degenerates along the line $[0, 1]$, then it is not difficult to see that μ_f does not have polynomial Fourier decay. On the other hand, [6, Theorem 1.1] implies that these pushforward measures have polynomial Fourier decay outside a very sparse set of exceptional frequencies, but we see that this cannot be upgraded to pointwise polynomial Fourier decay.

Remark 7.8. *Example 7.7 appears to contradict Mosquera and Olivo's [44, Theorem 3.1]. However, their proof required that the common linear part of the contractions has a non-trivial rotation, thus ruling out cases like Example 7.7.*

The next result shows that the obstructions we have described are essentially the only obstructions to polynomial Fourier decay for nonlinear, conjugate-to-linear, curve-reducible self-conformal measures.

Proposition 7.9. *Let* ν *be a nonlinear, curve*-reducible *self-conformal measure in* $\mathbb{C}$ *which is conjugate to linear and gives zero measure to straight lines. Then* ν *has polynomial Fourier decay unless there exists a self-similar measure* μ *supported in* $[0, 1]$ *such that*

$$\limsup_{\substack{\xi \in \mathbb{R} \\ |\xi| \to \infty}} \frac{|\widehat{\mu}(\xi)|}{|\xi|^{-\sigma}} = \infty \tag{7.2}$$

for all $\sigma > 0$*, some complex neighbourhood* U *of* $\operatorname{supp}(\mu)$*, and a conformal map* $g : U \to \mathbb{C}$ *which degenerates on* $[0, 1] \cap U$*, such that* $\nu = \mu_g$*.*

Proof. As in the proof of Theorem 7.6, we can write $\nu = \mu_f$ for a self-similar measure μ and holomorphic f defined on a neighbourhood U of $\operatorname{supp}(\mu)$. Since ν is curve-reducible, μ is affinely reducible and supported inside some line segment which after scaling and translation we may assume to be $[0, 1]$. Let V be any connected component of the domain of f which intersects $\operatorname{supp}(\mu)$. Since ν is nonlinear, f is nondegenerate on V, and since ν gives zero measure to lines, $f([0, 1] \cap V)$ is not contained in a line segment. If (7.2) fails, then this means that μ

[8]Incidentally, μ_f is also a self-affine measure for an IFS consisting of two affine contractions, with the same weights $(1/2, 1/2)$, see [6, Lemma 2.10]. The point is that the affine and conformal maps coincide for input values on the parabola, but are different away from the parabola.

has polynomial Fourier decay when thought of as a measure on $\mathbb{R}$, and by Theorem 3.1 (3), ν has polynomial Fourier decay. Alternatively, if f does not degenerate on $[0,1] \cap U$ then by Theorem 3.1 (2), ν once again has polynomial Fourier decay. This completes the proof. □

Theorem 7.6 and Proposition 7.9, show that if a characterisation of which self-similar measures on $\mathbb{C}$ have polynomial Fourier decay were to be obtained, then one would in fact have a characterisation for a much larger class of conjugate-to-linear self-conformal measures on $\mathbb{C}$. Unfortunately, however, such a characterisation seems far out of reach even for Bernoulli convolutions, which are a special class of self-similar measures.

7.2. **Self-conformal measures in higher dimensions.** In $\mathbb{R}^k$, $k \geqslant 3$, as mentioned in Section 2.6, conformal maps are Möbius transformations. In particular one can locally write a conformal map in the form

$$f(\mathbf{x}) = \mathbf{b} + \frac{\alpha A(\mathbf{x} - \mathbf{a})}{|\mathbf{x} - \mathbf{a}|^2}. \tag{7.3}$$

Lemma 7.10. *Let $k \geqslant 3$ and let $U \subseteq \mathbb{R}^k$ be non-empty, open and convex. Then every non-affine conformal map $f\colon U \subset \mathbb{R}^k \to \mathbb{R}^k$ is non-conical.*

Proof. Using the form (7.3), we see that $\mathbf{a} \notin U$, otherwise, f is singular at $\mathbf{a}$. Thus for any $\mathbf{v} \in \mathbb{S}^{k-1}$, if $|P_{\mathbf{v}}|$ is constant, it implies that either $f_{\mathbf{v}}$ is affine or proportional to the distant function $d(\cdot, \mathbf{a})$, see [59]. Without loss of generality, we take $\mathbf{a}, \mathbf{b} = \mathbf{0}$. Then

$$f(\mathbf{x}) = \frac{\alpha A\mathbf{x}}{|\mathbf{x}|^2}.$$

Thus, we see that

$$f_{\mathbf{v}}(\mathbf{x}) = \alpha |\mathbf{x}|^{-2} \langle A\mathbf{x}, \mathbf{v} \rangle,$$

which is never proportional to the distance function $d(\cdot, \mathbf{0})$, hence the result. □

We say that a self-conformal measure ν for an IFS $\{\varphi_i\}$ on $\mathbb{R}^k$ is conjugate to linear if there is a self-similar IFS $\{\psi_i\}$ and a conformal diffeomorphism f from an open neighbourhood U of the support of the self-similar set to an open neighbourhood of ν such that $\varphi_i = f \circ \psi_i \circ f^{-1}$ for all i. Using Theorem 3.1 (1) we can prove the following.

Theorem 7.11. *Fix an integer $k \geqslant 3$ and let ν be a self-conformal measure on $\mathbb{R}^k$. Assume that*

(1) ν is conjugate to linear,

(2) $\nu(V) = 0$ for all proper affine hyperplanes $V < \mathbb{R}^k$, and
(3) at least one of the conformal maps in the IFS is non-affine.

Then ν has polynomial Fourier decay.

Proof. As in Lemma 7.2, we can write $\nu = \mu_f$ for a self-similar measure μ on $\mathbb{R}^k$ and conformal map f. We may assume that each connected component of the domain of f intersects $\mathrm{supp}(\mu)$. By the nonlinearity assumption on ν and Liouville's theorem, on each such component we can write f as Möbius transformation, in other words a composition of translations, rotations, reflections, similarities, and inversions in $(k-1)$-spheres. More specifically, we can locally write

$$f(\mathbf{x}) = \mathbf{b} + \frac{\alpha A(\mathbf{x}-\mathbf{a})}{|\mathbf{x}-\mathbf{a}|^2},$$

for some scalar α, rotation A, and vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^k$. Since ν gives zero measure to affine hyperplanes, the smallest affine subspace of $\mathbb{R}^k$ containing $\mathrm{supp}(\mu)$ is either $\mathbb{R}^k$ itself or a $(k-1)$-dimensional subspace which is disjoint from each of the points $\mathbf{a}$. In the latter case ν is supported in a finite union of $(k-1)$-spheres, and we can write $\nu = \mu'_g$ for an analytic map g and affinely irreducible self-similar measure μ' on $\mathbb{R}^{k-1}$. In either case, Lemma 7.10 gives that $\mathbf{x} \mapsto |P_{\mathbf{v}}(\mathbf{x})|$ is never constant on any component, so Theorem 3.1 (1) (together with Fourier decay outside sparse frequencies from Theorem 3.9) gives polynomial Fourier decay for ν. ☐

8. Open problems

Theorem 1.3 being already complete as it is, there is still some room for further work, which we keep for potential future endeavours.

- *Beyond the non-expanding condition:* Theorem 3.1 with assumption (1) allows the self-similar measures to have arbitrary rotation group. The cost is that we have to assume f is non-conical. As mentioned in Remark 3.2 (1), we suspect that the weaker assumption of non-degeneracy should suffice, i.e. we still think that [11, Conjecture 5.14] should hold. If this is indeed the case, then there would in particular be polynomial Fourier decay for pushforwards of (possibly expanding) self-similar measures supported inside $\mathbb{R}^k \setminus \{\mathbf{0}\}$ for $k \geqslant 3$ under the map $\mathbf{x} \mapsto |\mathbf{x}|$ whose graph in $\mathbb{R}^{k+1}$ is a cone.

- *Relaxing the analyticity* f: We required that f is real analytic. The result is likely to hold for smooth maps with some conditions on its Taylor coefficients along the lines of the 'non-flatness' assumptions in [1, Section 1.4]. If one relaxes only assumes non-degeneracy / non-conicality as defined in Section 2.1 but relaxes the regularity from real-analytic to C^∞, then there is no hope of a general polynomial Fourier decay result even in the $k = d = 1$ case, as was recently shown in [9].
- *Quantifying the decay for more general pushforward maps:* In [11] we obtained nontrivial Fourier decay lower bounds for images of large self-similar measures under maps $f\colon \mathbb{R}^k \to \mathbb{R}$ with rigid curvature conditions. It is likely that one could obtain certain lower bounds for general non-degenerate maps given the 'degree of non-degeneracy' over the support of the to-be-pushed self-similar measure in place of the non-zero Gaussian curvature condition. Note that in the van der Corput type result [1, Theorem 1.1] for nonlinear images $\mathbb{R} \to \mathbb{R}$ of self-similar measures, the lower bound for the exponent of decay that is proved does depend on the 'degree of non-degeneracy' of the map. To prove such a result in higher dimensions, one cannot rely on the general non-quantitative version of the Fourier decay outside sparse frequencies property (Definition 3.8).
- *Nonlinear self-conformal measures supported in curves or in higher dimensions:* In Proposition 7.9 we addressed curve-reducible self-conformal measures which are conjugate to linear, but this leaves open the question of those which are not conjugate to linear. This may require quantitative versions of the arguments from [4, 12].

Conjecture 8.1. *Every self-conformal measure on $\mathbb{C}$ which is not conjugate to linear and which is supported in an analytic curve that is not a line has polynomial Fourier decay.*

Regarding self-conformal measures in $\mathbb{R}^k$ for $k \geqslant 3$, the following seems plausible.

Conjecture 8.2. *Theorem 7.11 remains true even without the assumption that ν is conjugate to linear.*

Recall that [5] is about planar measures. Conjecture 8.2 would perhaps require challenging spectral gap arguments (noting [5, 10]). One could also consider Fourier decay for Gibbs measures for C^1 potentials, associated with conjugate-to-linear conformal IFSs in $\mathbb{R}^k$; in the non-conjugate case Gibbs measures were considered in [10, 29].

- *Nonlinear fibres and conditional measures:* In [11, Section 4], we discussed several results related to nonlinear projections of self-similar measures that can be treated with our Fourier decay results. However, we note here that our arguments do not give much information about fibres or conditional measures for those projections. More precisely, let μ be a probability measure on $\mathbb{R}^k$ and let $f\colon \mathbb{R}^k \to \mathbb{R}^d$ be a smooth (or analytic) map. We considered several properties of the measure μ_f in the case when μ is self-similar. Now, let $\mu^{\mathbf{x}}$ for $\mathbf{x} \in \mathbb{R}^d$ be the disintegration of μ against μ_f. For μ_f-a.e. $\mathbf{x}$, the conditional measure is a well-defined probability measure. Several questions can be asked; we list two of them.
- What can be said about $\dim(\mathrm{supp}(\mu^{\mathbf{x}}))$ for μ_f-generic $\mathbf{x}$ and for specific $\mathbf{x}$? Here dim can be $\dim_{\mathrm{H}}, \dim_{\mathrm{B}}$, etc. It is not hard to obtain some upper bound for $\dim(\mathrm{supp}(\mu^{\mathbf{x}}))$ using results from [17, 53, 62]. However, much less is known about the lower bound.
- What can be said about the exact dimensionality of $\mu^{\mathbf{x}}$ for μ_f-generic $\mathbf{x}$ and for specific $\mathbf{x}$? Here, we say that a probability measure ν on $\mathbb{R}^k$ is exact dimensional if there is some $s \geqslant 0$ so that for ν-a.e. $\mathbf{x} \in \mathbb{R}^k$,
$$\lim_{\delta\to 0} \frac{\log \nu(B_\delta(\mathbf{x}))}{\log \delta} = s.$$
It is known that all self-conformal [25] and self-affine [24] measures are exact dimensional. However, much less is known about the exact dimensionality of conditional measures under smooth projections.

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