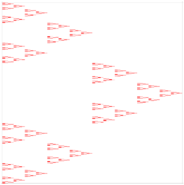
Intermediate dimensions of Bedford–McMullen carpets

Amlan Banaji

University of St Andrews



Based on work in 'Intermediate dimensions of Bedford-McMullen carpets with applications to Lipschitz equivalence' (with István Kolossváry), arXiv preprint (2021), https://arxiv.org/abs/2111.05625

I also thank István for many of the pictures in these slides.

Iterated function systems

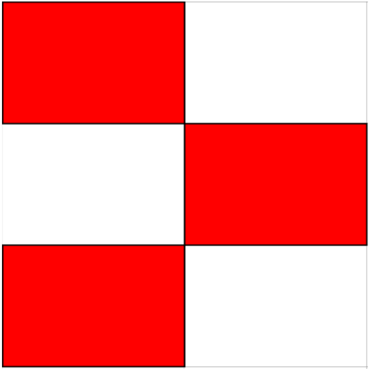
- For fractal sets, the Hausdorff dimension is sometimes smaller than the box dimension, indicating an inhomogeneity in space.
- An iterated function system (IFS) is a finite set of contractions $\{S_i \colon X \to X\}_{i \in I}$ where $X \subset \mathbb{R}^d$ is closed. Hutchinson (1981) showed there is a unique non-empty compact attractor satisfying

$$F = \cup_{i \in I} S_i(F).$$

- Self-similar sets have equal Hausdorff and box dimensions.
- Planar self-affine sets which satisfy separation conditions "typically" have equal Hausdorff and box dimensions (Bárány-Hochman-Rapaport, '19), but not always...

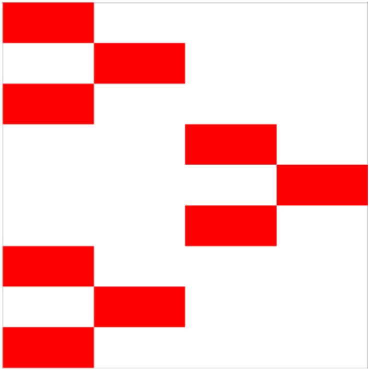
Bedford–McMullen carpets (Bedford '84, McMullen '84)

- A widely-studied class of self-affine fractals in the plane.
- Divide a square into an $m \times n$ grid, $m < n$. Write $\gamma := \frac{\log n}{\log m}$. Parameters: $M :=$ #non-empty columns, $N_{\hat{\imath}} :=$ #maps in column $\hat{\imath}$, $N := N_1 + \ldots + N_M$.
- IFS: $\{S_1, \ldots, S_N\}$ where $S_i(\underline{x}) := \begin{pmatrix} 1/m & 0 \\ 0 & 1/n \end{pmatrix} (\underline{x}) + \underline{t}_i$.



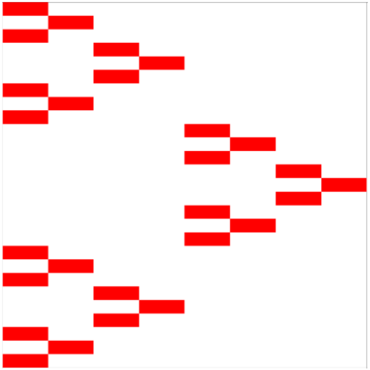
Bedford–McMullen carpets (Bedford '84, McMullen '84)

- A widely-studied class of self-affine fractals in the plane.
- Divide a square into an $m \times n$ grid, $m < n$. Write $\gamma := \frac{\log n}{\log m}$. Parameters: $M :=$ #non-empty columns, $N_{\hat{\imath}} :=$ #maps in column $\hat{\imath}$, $N := N_1 + \ldots + N_M$.
- IFS: $\{S_1, \ldots, S_M\}$ where $S_i(\underline{x}) := \begin{pmatrix} 1/m & 0 \\ 0 & 1/n \end{pmatrix} (\underline{x}) + \underline{t}_i$.



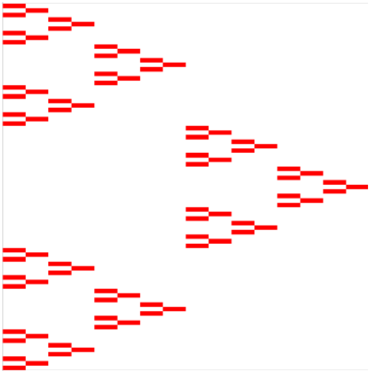
Bedford–McMullen carpets (Bedford '84, McMullen '84)

- A widely-studied class of self-affine fractals in the plane.
- Divide a square into an $m \times n$ grid, $m < n$. Write $\gamma := \frac{\log n}{\log m}$. Parameters: $M :=$ #non-empty columns, $N_{\hat{\imath}} :=$ #maps in column $\hat{\imath}$, $N := N_1 + \ldots + N_M$.
- IFS: $\{S_1, \ldots, S_M\}$ where $S_i(\underline{x}) := \begin{pmatrix} 1/m & 0 \\ 0 & 1/n \end{pmatrix} (\underline{x}) + \underline{t}_i$.



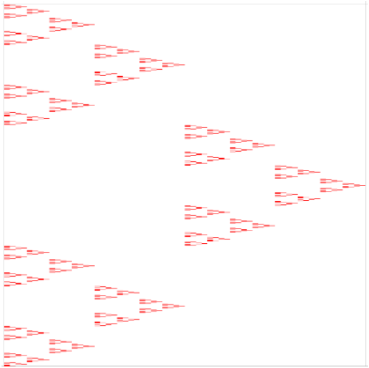
Bedford–McMullen carpets (Bedford '84, McMullen '84)

- A widely-studied class of self-affine fractals in the plane.
- Divide a square into an $m \times n$ grid, $m < n$. Write $\gamma := \frac{\log n}{\log m}$. Parameters: $M :=$ #non-empty columns, $N_{\hat{\imath}} :=$ #maps in column $\hat{\imath}$, $N := N_1 + \ldots + N_M$.
- IFS: $\{S_1, \ldots, S_M\}$ where $S_i(\underline{x}) := \begin{pmatrix} 1/m & 0 \\ 0 & 1/n \end{pmatrix} (\underline{x}) + \underline{t}_i$.



Bedford–McMullen carpets (Bedford '84, McMullen '84)

- A widely-studied class of self-affine fractals in the plane.
- Divide a square into an $m \times n$ grid, $m < n$. Write $\gamma := \frac{\log n}{\log m}$. Parameters: $M :=$ #non-empty columns, $N_{\hat{\imath}} :=$ #maps in column $\hat{\imath}$, $N := N_1 + \ldots + N_M$.
- IFS: $\{S_1, \ldots, S_M\}$ where $S_i(\underline{x}) := \begin{pmatrix} 1/m & 0 \\ 0 & 1/n \end{pmatrix} (\underline{x}) + \underline{t}_i$.



Hausdorff and box dimensions

Cylinders become elongated and line up, causing inhomogeneity in space.

Theorem (Bedford '84, McMullen '84)

$$\dim_{\mathrm{H}} \Lambda = \frac{1}{\log m} \log \left(\sum_{\hat{\imath}=1}^{M} N_{\hat{\imath}}^{\frac{\log m}{\log n}} \right); \qquad \dim_{\mathrm{B}} \Lambda = \frac{\log M}{\log m} + \frac{\log(N/M)}{\log n}.$$

In particular, $\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} \Lambda$ if and only if Λ has uniform vertical fibres: $N_{\hat{\imath}} = N/M$ for $\hat{\imath} = 1, \ldots, M$. Throughout we assume this is not the case.

Hausdorff and box dimensions

- (Upper) box dimension of a non-empty, bounded $F \subset \mathbb{R}^d$:

$$\overline{\dim}_{\mathrm{B}} F = \inf\{\, s \geq 0 : \text{for all } \epsilon > 0 \text{ there exists } \delta_0 \in (0,1] \text{ such that for all } \delta \in (0,\delta_0) \text{ there exists a cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } |U_i| = \delta \text{ for all } i, \text{ and } \sum_i |U_i|^s \leq \epsilon \,\},$$

where $|U_i| = \operatorname{diam}(U_i)$.

- Hausdorff dimension:

$$\dim_{\mathrm{H}} F = \inf\{\, s \geq 0 : \text{for all } \epsilon > 0 \text{ there exists a finite or countable cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } \sum_i |U_i|^s \leq \epsilon \,\}$$

- Falconer, Fraser and Kempton (2020) noted that these “may be regarded as two extreme cases of the same definition...”

Intermediate dimensions

and defined the upper θ-intermediate dimension of F for $\theta \in (0,1)$ by

$$\overline{\dim}_\theta F = \inf\{\, s \geq 0 : \text{for all } \epsilon > 0 \text{ there exists } \delta_0 \in (0,1] \text{ such that for all } \delta \in (0,\delta_0) \text{ there exists a cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } \delta^{1/\theta} \leq |U_i| \leq \delta \text{ for all } i, \text{ and } \sum_i |U_i|^s \leq \epsilon \,\}.$$

Indeed for all sets F, $\dim_{\mathrm{H}} F \leq \overline{\dim}_\theta F \leq \overline{\dim}_{\mathrm{B}} F$ for all $\theta \in (0,1)$. Function $\theta \mapsto \overline{\dim}_\theta F$ increasing, continuous for $\theta \in (0,1]$ but not necessarily at $\theta = 0$ (see Φ-intermediate dimensions, B. '20).
Example of dimension interpolation. See also Assouad spectrum (Fraser–Yu, '18).

Asymptotic behaviour near $\theta = 0$

Let Λ be a Bedford–McMullen carpet with non-uniform vertical fibres.

Proposition (Falconer, Fraser and Kempton, '20)

For small enough θ,

$$\overline{\dim}_\theta \Lambda \leq \dim_{\mathrm{H}} \Lambda + \frac{C}{-\log \theta}.$$

In particular, $\overline{\dim}_\theta \Lambda$ is continuous at $\theta = 0$.

Proposition (B.–Kolossváry, '21)

For small enough θ,

$$\dim_{\mathrm{H}} \Lambda + \frac{C_1}{(\log \theta)^2} \leq \dim_\theta \Lambda \leq \dim_{\mathrm{H}} \Lambda + \frac{C_2}{(\log \theta)^2}.$$

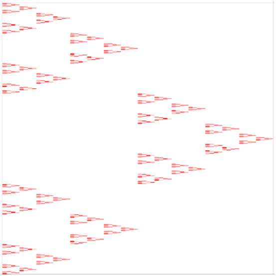
In particular, the slope of $\theta \mapsto \dim_\theta \Lambda$ tends to ∞ as $\theta \to 0$.

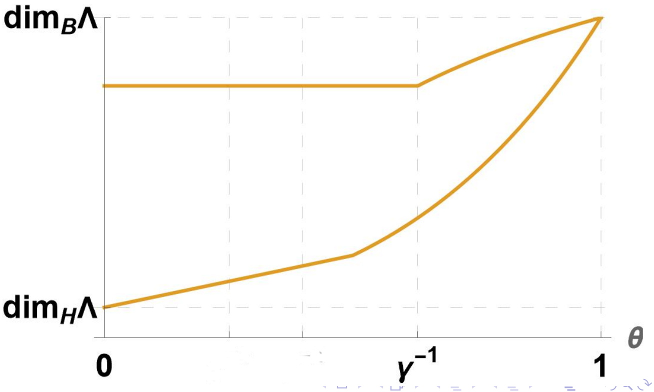
Consequences of continuity at $\theta = 0$

- If $\dim_{\mathrm{H}} \Lambda < 1 \leq \dim_{\mathrm{B}} \Lambda$ then $\overline{\dim}_{\mathrm{B}} \pi(\Lambda) < 1$ for every orthogonal projection π from $\mathbb{R}^2$ onto a 1-dimensional subspace (Burrell–Falconer–Fraser, '21).
- If $B_h \colon \mathbb{R}^2 \to \mathbb{R}^2$ is index-h fractional Brownian motion, then if $h > (\dim_{\mathrm{H}} \Lambda)/2$ then almost surely $\overline{\dim}_{\mathrm{B}} B_h(\Lambda) < 2$ (Burrell, '20).

Further bounds

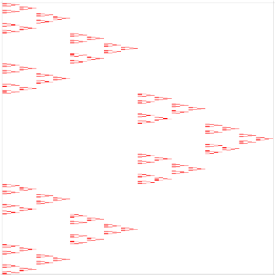
- Falconer, Fraser and Kempton ('20) proved a lower bound showing that $\dim_\theta \Lambda > \dim_{\mathrm{H}} \Lambda$ for all $\theta > 0$.
- Kolossváry ('20) proved upper and lower bounds near $\theta = 1$ showing that $\dim_\theta \Lambda < \dim_{\mathrm{B}} \Lambda$ for all $\theta < 1$, and that the graph is neither convex nor concave in general.
- A general lower bound near $\theta = 1$ was obtained in B. ('20).

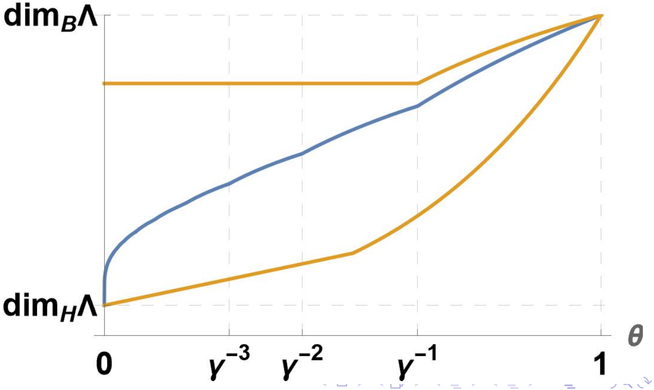




Further bounds

- Falconer, Fraser and Kempton ('20) proved a lower bound showing that $\dim_\theta \Lambda > \dim_{\mathrm{H}} \Lambda$ for all $\theta > 0$.
- Kolossváry ('20) proved upper and lower bounds near $\theta = 1$ showing that $\dim_\theta \Lambda < \dim_{\mathrm{B}} \Lambda$ for all $\theta < 1$, and that the graph is neither convex nor concave in general.
- A general lower bound near $\theta = 1$ was obtained in B. ('20).





Rate function

- Define the function $I(t)$ as the Legendre transform

$$I(t) := \sup_{\lambda \in \mathbb{R}} \left\{ \lambda t - \log \left(\frac{1}{M} \sum_{\hat{\jmath}=1}^{M} N_{\hat{\jmath}}^{\lambda} \right) \right\}.$$

  This is a large deviation rate function, describing the exponential decline of the probability of certain extreme events - see Cramér's theorem. It is strictly convex.

- For $s \in \mathbb{R}$, define the function $T_s \colon \mathbb{R} \to \mathbb{R}$ by

$$T_s(t) := \left(s - \frac{\log M}{\log m} \right) \log n + \gamma I(t).$$

- For $\ell \in \mathbb{N}$, write $T_s^{\ell} := \underbrace{T_s \circ \cdots \circ T_s}_{\ell \text{ times}}$, and T_s^0 is the identity. Define

$$t_\ell(s) := T_s^{\ell-1} \left(\left(s - \frac{\log M}{\log m} \right) \log n \right).$$

Main result: formula for the intermediate dimensions

Theorem (B.–Kolossváry, '21)

Let Λ be any Bedford–McMullen carpet with non-uniform vertical fibres. For fixed $\theta \in (0,1)$ let $L = L(\theta) \in \mathbb{N}$ be such that $\gamma^{-L} < \theta \leq \gamma^{-(L-1)}$. Then there exists a unique solution $s = s(\theta) \in (\dim_{\mathrm{H}} \Lambda, \dim_{\mathrm{B}} \Lambda)$ to the equation

$$\gamma^L \theta \log N - (\gamma^L \theta - 1) t_L(s) + \gamma(1 - \gamma^{L-1}\theta)(\log M - I(t_L(s))) - s \log n = 0,$$

and $s(\theta) = \dim_\theta \Lambda$.

In particular the intermediate dimensions exist.

If the carpet has just two column types then we can calculate the rate function explicitly. We can always compute the intermediate dimensions numerically and draw plots.

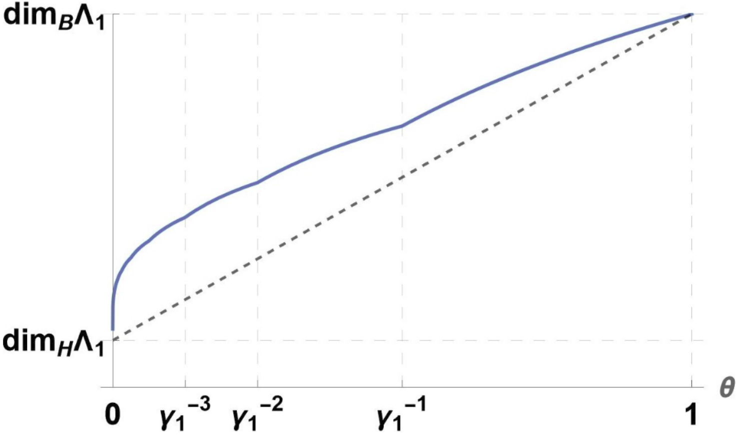
Comments about the proof of the upper bound

- Upper bound involves the explicit construction of an intricate cover using scales $\delta, \delta^{\gamma}, \delta^{\gamma^2}, \ldots, \delta^{\gamma^{L-1}}$ and $\delta^{1/\theta}, \delta^{1/(\gamma\theta)}, \ldots, \delta^{1/(\gamma^{L-1}\theta)}$. This is enough to achieve the intermediate dimensions as the exponent, but to achieve the true smallest possible s-cost (up to multiplicative constants) one would have to use a number of scales tending to ∞ as $\delta \to 0$, without leaving any exponential gap between scales used.
- The proof simplifies substantially when $\theta \geq 1/\gamma$ (use just largest and smallest scales) or $\theta = \gamma^{-k}$ (use scales $\delta, \delta^{\gamma}, \ldots, \delta^{\gamma^k}$).
- We break the carpet into approximate squares and decide which parts of each approximate square to cover at which scale depending on how the different parts of the symbolic representation of the approximate square relate to each other.
- Significance of γ: a cylinder of length δ has height $\approx \delta^{\gamma}$. An approximate square of size δ is made of stacking such cylinders on top of each other.
- These are the first sets for which we can prove that we need to use more than two scales to achieve the intermediate dimension (when θ is small). Though the cover for continued fraction sets and other infinitely-generated self-conformal sets also uses many scales (B.–Fraser, '21).

Tools for the proof

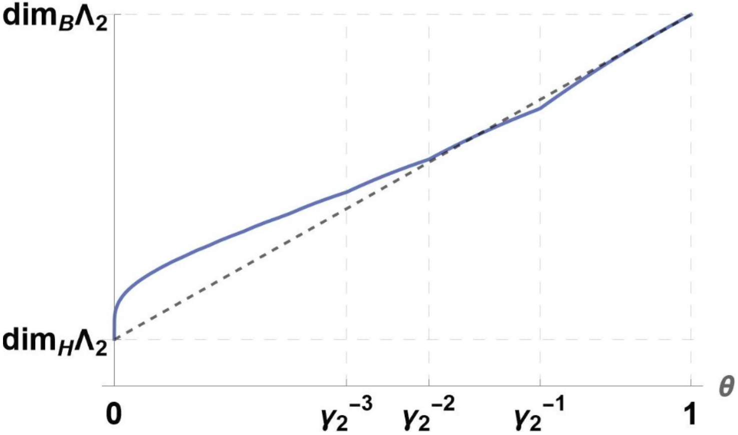
- Lower bound uses a variant of a mass distribution principle proved by Falconer, Fraser and Kempton ('20).
- Throughout, we use the method of types, also used by Kolossváry ('21) to calculate the box dimensions of certain higher-dimensional self-affine sponges.

Varying the parameters can lead to the graph having different shapes.



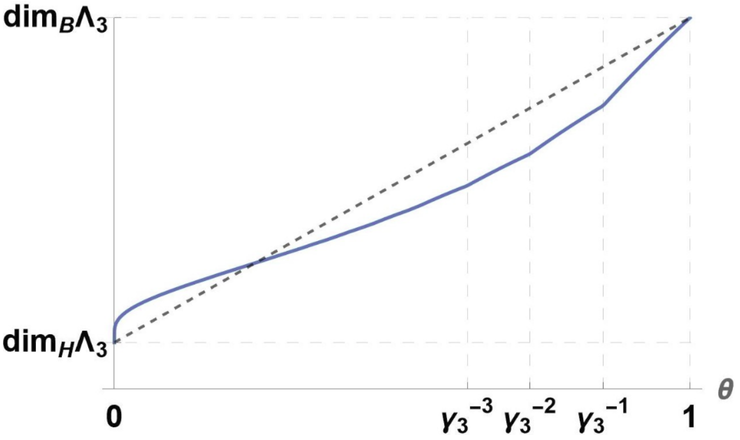
Shape of the graph

Varying the parameters can lead to the graph having different shapes.



Shape of the graph

Varying the parameters can lead to the graph having different shapes.



Properties of the graph: a form not seen in previous examples

- Using implicit differentiation shows that the intermediate dimensions are strictly increasing (with a lower bound on the slope independent of θ).
- At every negative integer power of γ there is a phase transition (derivative jumps up). Intuitively, this is because there is a discrete jump in the number of scales used.
- Differentiating again and using properties of the rate function shows the graph is strictly concave between phase transitions.
- The graph is real analytic between phase transitions. Indeed, the rate function is analytic as the Legendre transform of an analytic function. Therefore the function of two variables is analytic along arbitrary line segments. So by a result of Siciak ('70), the function is jointly analytic in θ and s. By the analytic version of the implicit function theorem, the zero set ($\dim_\theta \Lambda$) is analytic.
- To prove $\dim_\theta \Lambda \approx \dim_{\mathrm{H}} \Lambda + \frac{\text{const}}{(\log \theta)^2}$ near $\theta = 0$ we estimate $L(\theta)$ (which is the number of the $t_i(\dim_\theta \Lambda)$) in terms of $\dim_\theta \Lambda$ using Taylor's theorem, using the fact that the t_i cluster around a neutral fixed point of the function $T_{\dim_{\mathrm{H}} \Lambda}$.

Multifractal analysis

- Let ν be the uniform Bernoulli measure supported on a Bedford–McMullen carpet giving mass $1/N$ to each first-level cylinder, satisfying

$$\nu(A) = \sum_{i=1}^{N} \frac{1}{N} \nu(S_i^{-1} A) \text{ for all Borel sets } A \subset \mathbb{R}^2.$$

- Kenyon and Peres ('96) showed that ν is exact dimensional: the local dimension

$$\dim_{\mathrm{loc}}(\nu, x) = \lim_{r \to 0} \frac{\log \nu(B(x,r))}{\log r}$$

exists and is constant at ν-almost every $x \in \Lambda$.
- The (fine/Hausdorff) multifractal spectrum of ν,

$$f_\nu(\alpha) := \dim_{\mathrm{H}}\{x \in \operatorname{supp} \nu : \dim_{\mathrm{loc}}(\nu, x) = \alpha\},$$

has been computed by Jordan and Rams ('11), building on work of King ('95) and Barral and Mensi ('07). In fact, using properties of Legendre transforms, we see that it is directly related to the rate function:

$$I(t) = -(\log m) f_\nu \left(\frac{\log N}{\log m} - \left(\frac{1}{\log m} - \frac{1}{\log n} \right) t \right) + \frac{t}{\gamma} + \log M.$$

Connection between intermediate dimensions and multifractal analysis

Theorem (B.–Kolossváry, '21)

If Λ, Λ' are Bedford–McMullen carpets with non-uniform vertical fibres, then the intermediate dimensions are equal for all θ if and only if the corresponding uniform Bernoulli measures have the same multifractal spectrum.
In this case the carpets can be defined on the same grid, and if $m \times n$ and $m' \times n'$ are any grids on which the respective carpets can be defined, then $\frac{\log n}{\log n'} = \frac{\log m}{\log m'} \in \mathbb{Q}$.

- Now assume the carpets are defined on the same $m \times n$ grid to begin with. Then equality of the multifractal spectra was shown by Rao, Yang and Zhang ('20) to be equivalent to an explicit condition on the parameters of the carpets. In particular, if the number of non-empty columns is equal, then this condition says that the column sequence of one carpet must be a permutation of the other.
- We show that for carpets on the same grid, $\dim_\theta \Lambda = \dim_\theta \Lambda'$ for all $\theta \in (0,1)$ if and only if $\dim_\theta \Lambda = \dim_\theta \Lambda'$ for every θ in an open interval $(a,b) \subset [\gamma^{-1}, 1]$.

Examples

There exist two carpets, taken from an example of Rao, Yang and Zhang ('20), which are defined on the same grid with different column sequences and a different number of non-empty columns but have the same intermediate dimensions.
We also construct two carpets on two different grids which have the same intermediate dimensions on an interval $(\theta_0, 1)$ but not for the whole spectrum of θ.

Bi-Lipschitz equivalence

A difficult open problem: find an explicit necessary and sufficient condition that determines, given two iterated function systems each generating a Bedford–McMullen carpet, whether or not the two carpets are bi-Lipschitz equivalent. Partial progress by Li, Li and Miao ('07), Yang and Zhang ('20), Rao, Yang and Zhang ('20).

Since the intermediate dimensions can easily be seen to be bi-Lipschitz invariant, we obtain the following necessary condition for bi-Lipschitz equivalence as a consequence of the previous theorem:

Corollary

Suppose two Bedford–McMullen carpets with non-uniform vertical fibres are bi-Lipschitz equivalent. Then their uniform Bernoulli measures have the same multifractal spectra, and both carpets can be defined on the same $m \times n$ grid. Moreover, if $m_1 \times n_1$ and $m_2 \times n_2$ are any grids on which the respective carpets can be defined, then $\frac{\log m_1}{\log m_2} = \frac{\log n_1}{\log n_2} \in \mathbb{Q}$.

Improves a result of Rao, Yang and Zhang ('20) where the carpets are assumed to be totally disconnected and defined on the same grid.

Bi-Lipschitz equivalence

In particular if a carpet can be defined on two grids $m \times n$ and $m' \times n'$ then $\frac{\log m}{\log m'} = \frac{\log n}{\log n'} \in \mathbb{Q}$ (the equality was noted by Fraser and Yu ('18) using the Assouad spectrum, and the rationality is related to a result of Meiri–Peres (1999) related to Furstenberg's Times 2, Times 3 conjectures/theorems).
Consider the two carpets with $m = M = 32$, $n = 243$, and parameters

$$\Lambda_1: \qquad N_1 = N_2 = 27, \quad N_3 = \cdots = N_{13} = 3, \quad N_{14} = \cdots = N_{32} = 1,$$
$$\Lambda_2: \qquad N_1 = 27, \quad N_2 = \cdots = N_7 = 9, \quad N_8 = \cdots = N_{32} = 1.$$

Then by the discussion above they have different intermediate dimensions. But all other (usual) dimensions are equal. If the maps are arranged so the carpets are not totally disconnected then the Rao–Yang–Zhang result does not apply, and only the intermediate dimensions can tell that they are not bi-Lipschitz equivalent.

Hölder distortion

- Can do better than just saying they are not bi-Lipschitz equivalent. Straightforward to see that if $f\colon \Lambda' \to \mathbb{R}^2$ is α-Hölder then $\dim_\theta f(\Lambda') \le \alpha^{-1} \dim_\theta \Lambda'$.
- So if $f(\Lambda') \supseteq \Lambda$ then using the optimal value $\theta = \gamma^{-2} = \left(\frac{\log 2}{\log 3}\right)^2 \approx 0.40$,

$$\alpha \le \frac{\dim_{\gamma^{-2}} \Lambda'}{\dim_{\gamma^{-2}} f(\Lambda')} \le \frac{\dim_{\gamma^{-2}} \Lambda'}{\dim_{\gamma^{-2}} \Lambda} < 0.9995,$$

with the last inequality computed numerically.

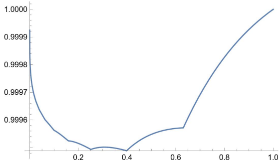


Figure: Ratio of intermediate dimensions

Intermediate dimensions can also sometimes give better Hölder distortion estimates than the Hausdorff or box dimensions for continued fraction sets (B.–Fraser, '21).

Open problems

- Can one make further progress towards characterising when two Bedford–McMullen carpets are bi-Lipschitz equivalent? In particular, do there exist two carpets with non-uniform vertical fibres defined on the same grid with different column sequences that are bi-Lipschitz equivalent?
- What can be said about the intermediate dimensions of other classes of self-affine sets such as higher-dimensional Bedford–McMullen sponges, or Lalley–Gatzouras or Barański carpets/sponges? Perhaps would require some similar techniques, but with substantial additional complications?

Thank you for listening! Questions welcome.