Intermediate dimensions of Bedford–McMullen carpets

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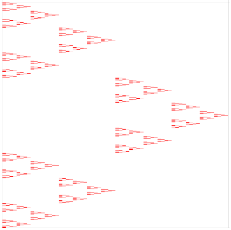


[1]Based on joint work with István Kolossváry in https://arxiv.org/abs/2111.05625

Intermediate dimensions

- Box dimension: cover a set with balls of the same size.
- Hausdorff dimension: sets in the cover can have different sizes:

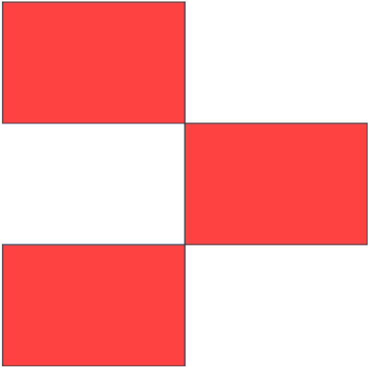
$$\dim_{\mathrm{H}} F \leq \overline{\dim}_{\mathrm{B}} F$$

- Intermediate dimensions (Falconer, Fraser and Kempton, '20) for $\theta \in (0,1)$:

$$\overline{\dim}_\theta F = \inf\{\, s \geq 0 : \text{for all } \epsilon > 0 \text{ there exists } \delta_0 \in (0,1] \text{ such that for all } \delta \in (0,\delta_0) \text{ there exists a cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } \delta^{1/\theta} \leq |U_i| \leq \delta \text{ for all } i, \text{ and } \sum_i |U_i|^s \leq \epsilon \,\}.$$

Bedford–McMullen carpets

Divide a square into an $m \times n$ grid, $2 \leq m < n$.



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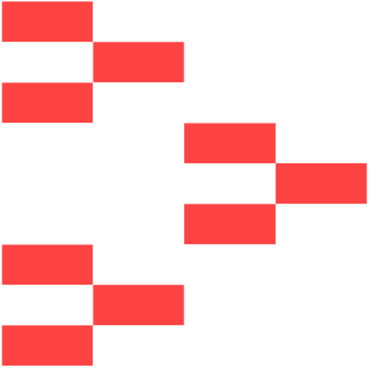

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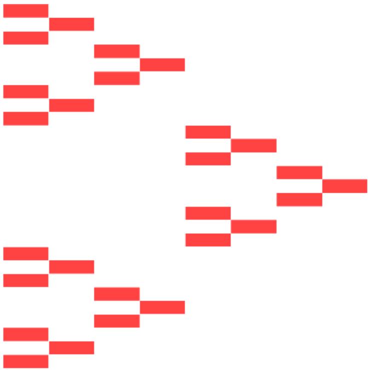

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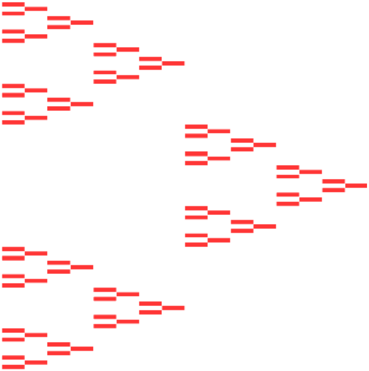

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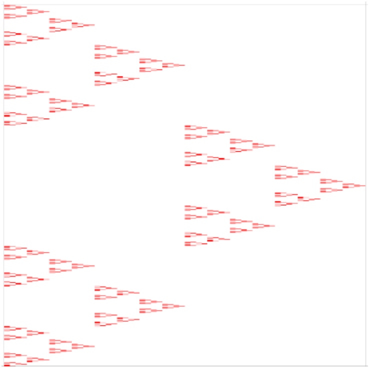


Hausdorff and box dimensions

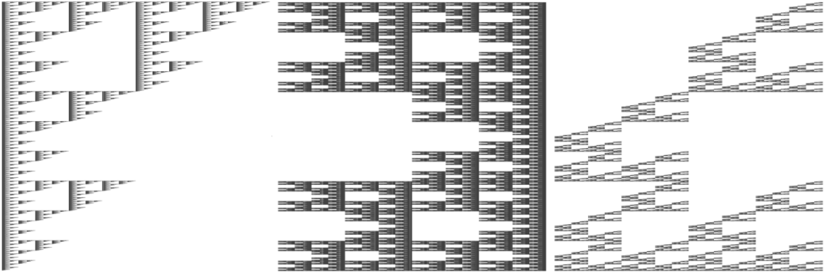
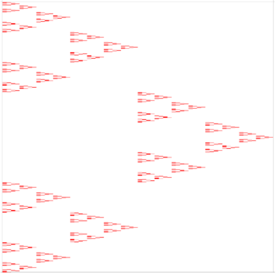


Figure: Three different Bedford–McMullen carpets, by Jonathan Fraser

Bedford ('84) and McMullen ('84) independently calculated their Hausdorff and box dimensions.
Throughout, we assume that Λ has non-uniform vertical fibres, or equivalently that $\dim_{\mathrm{H}} \Lambda < \dim_{\mathrm{B}} \Lambda$.

Graph of the intermediate dimensions



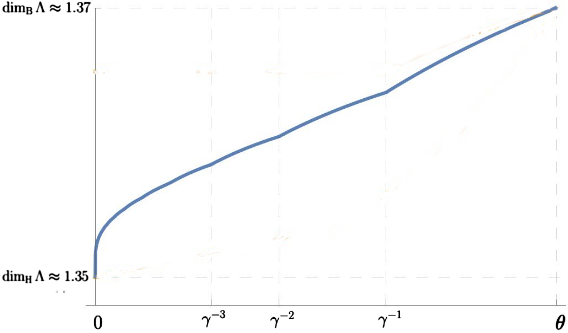
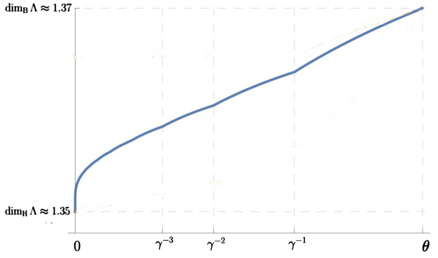


Figure: Here, $\gamma := \log n / \log m$

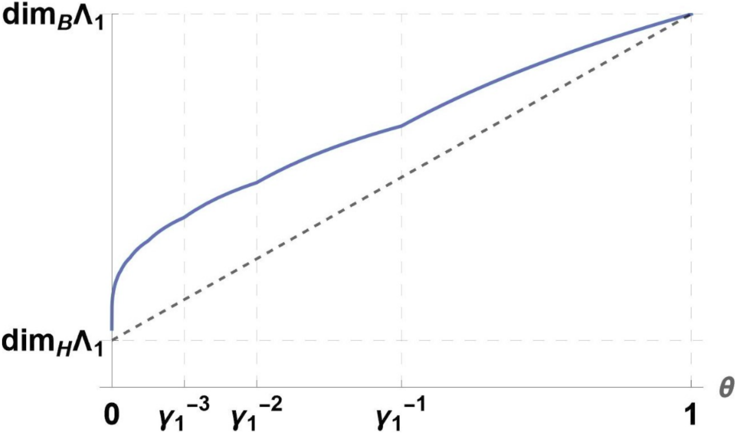
Proof involves constructing a cover using an increasing number of scales as $\theta \to 0$.

Intermediate dimensions of Bedford–McMullen carpets

- Phase transitions at negative integer powers of $\log n / \log m$.
- Real analytic and strictly concave between phase transitions
- Strictly increasing
- Right derivative tends to ∞ as $\theta \to 0$
- Continuous for $\theta \in [0, 1]$ (Falconer–Fraser–Kempton, '20)



Different possible shapes of the graph



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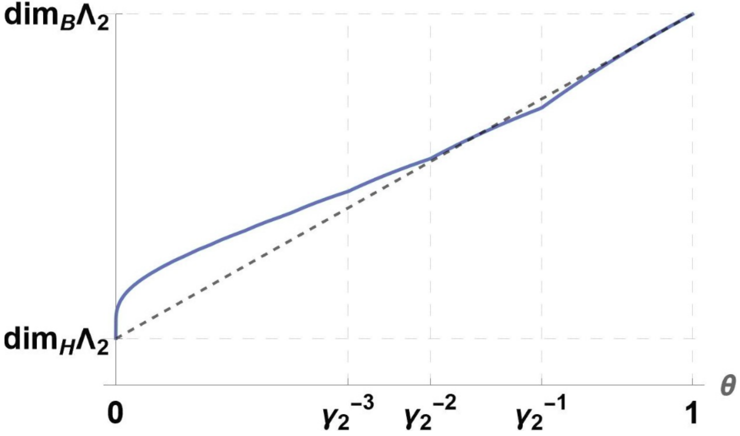

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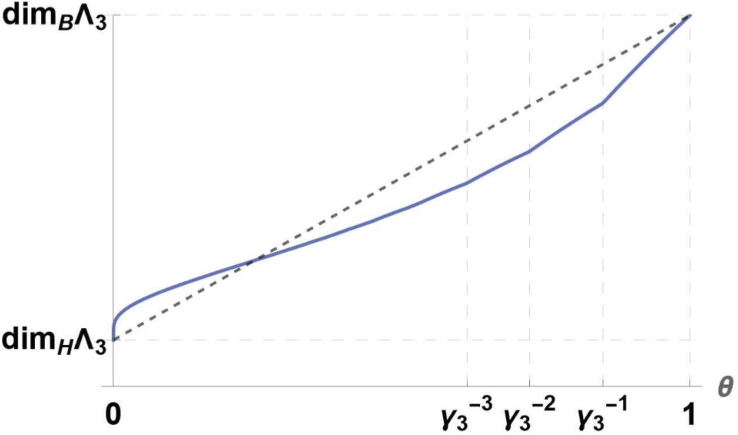

Multifractal analysis

- Let ν be the uniform Bernoulli measure supported on a Bedford–McMullen carpet, satisfying

$$\nu(A) = \sum_{i=1}^{N} \frac{1}{N} \nu(S_i^{-1} A) \text{ for all Borel sets } A \subset \mathbb{R}^2.$$

  where N is the total number of contractions.
- Jordan and Rams ('11) computed the multifractal spectrum of ν,

$$f_\nu(\alpha) \coloneqq \dim_{\mathrm{H}} \{ x \in \operatorname{supp} \nu : \dim_{\mathrm{loc}}(\nu, x) = \alpha \},$$

  building on work of King ('95).

Theorem (B.–Kolossváry, '21+)

If Λ, Λ' are Bedford–McMullen carpets with non-uniform vertical fibres, then the intermediate dimensions are equal for all θ if and only if the corresponding uniform Bernoulli measures have the same multifractal spectra.

Bi-Lipschitz equivalence

If $f\colon \Lambda \to \Lambda'$ is bi-Lipschitz then $\dim_\theta \Lambda = \dim_\theta \Lambda'$ for all θ.

Corollary

If carpets Λ and Λ' with non-uniform vertical fibres are bi-Lipschitz equivalent then their uniform Bernoulli measures have the same multifractal spectra.

This improves a result of Rao, Yang and Zhang ('21+).

Thank you for listening!

Please see my poster for more details and to ask any questions!