Dimensions of infinitely generated self-conformal sets

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[1] Based on work in:
- 'Intermediate dimensions of infinitely generated attractors' (with Jonathan M. Fraser), **Trans. Amer. Math. Soc.** (to appear), `https://arxiv.org/abs/2104.15133`, 2021.
- 'Assouad type dimensions of infinitely generated self-conformal sets' (with Jonathan M. Fraser), Preprint, `https://arxiv.org/abs/2207.11611`, 2022.

Iterated function systems (IFSs)

- Let $X \subset \mathbb{R}^n$ be compact and let $\{S_i \colon X \to X\}_{i \in I}$ be a finite IFS satisfying the open set condition (OSC). There is a unique non-empty compact attractor F satisfying $F = \bigcup_{i \in I} S_i(F)$.
- If each S_i is assumed to be a similarity map with contraction ratio c_i then the Hausdorff, box and Assouad dimensions of F coincide with the unique $h \geq 0$ satisfying Hutchinson's formula

$$\sum_{i \in I} c_i^h = 1.$$

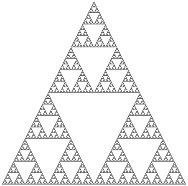

Figure: The Sierpinski gasket has Hausdorff dimension $\log 3 / \log 2$.

- These dimensions coincide even if the contractions are only assumed to be conformal.

Infinite conformal iterated function systems (Mauldin–Urbański, '96, Proc. LMS)

An (infinite) conformal iterated function system (CIFS) is a countable family of maps $\{S_i\colon X \to X\}_{i \in I}$ that satisfies the following properties:

- Uniform contraction
- Conformality: each S_i extends to a conformal map from a bounded open set $V \to V$
- Open set condition: $\mathrm{Int}(X) \neq \varnothing$ and $\bigcup_{i \in I} S_i(\mathrm{Int}(X)) \subseteq \mathrm{Int}(X)$ with the union disjoint.
- Cone condition
- Bounded distortion property

Limit set

The limit set F of a CIFS can be defined as the largest set (by inclusion) which satisfies

$$F = \bigcup_{i \in I} S_i(F).$$

It is non-empty but is not generally closed.

First and second level cylinders for an infinitely generated self-similar set

Pressure function

- For $w \in I^k$ define

$$R_w := \sup_{x,y \in X, x \neq y} \frac{||S_w(x) - S_w(y)||}{||x - y||},$$

  the smallest possible Lipschitz constant for S_w.
- We can define the topological pressure function $p\colon (0,\infty) \to [-\infty, \infty]$ by

$$p(t) := \lim_{k \to \infty} \frac{1}{k} \log \sum_{w \in I_k} R_w^t.$$

Hausdorff and box dimensions

Henceforth, F will be the limit set of a CIFS.

Theorem (Mauldin–Urbański, '96)

$\dim_{\mathrm{H}} F = \inf\{\, t > 0 : p(t) < 0 \,\}$

- In particular, if each S_i is a similarity with contraction ratio c_i then $\dim_{\mathrm{H}} F = \inf\{\, t \geq 0 : \sum_{i \in I} c_i^t \leq 1 \,\}$ (there may not exist $t \geq 0$ such that $p(t) = 0$).

Here and later, fix $x \in X$ and write $P := \{\, S_i(x) : i \in I \,\}$:

Theorem (Mauldin–Urbański, '99, TAMS)

$\overline{\dim}_{\mathrm{B}} F = \max\{\dim_{\mathrm{H}} F, \overline{\dim}_{\mathrm{B}} P\}$

Assouad dimension

Assouad dimension captures the scaling behaviour of the 'thickest' parts of the set, and has applications to embeddability problems.

$$\dim_{\mathrm{A}} F := \inf\{\, \alpha : \text{ there exists } C > 0 \text{ such that for all } x \in F \text{ and } 0 < r < R, \text{ we have } N_r(B(x,R) \cap F) \leq C(R/r)^{\alpha} \,\}.$$

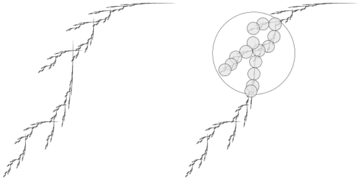

Figure: Covering a ball for the Assouad dimension. Picture by Jonathan Fraser.

In general, $\dim_{\mathrm{H}} F \leq \overline{\dim}_{\mathrm{B}} F \leq \dim_{\mathrm{A}} F$.

Theorem (B.–Fraser, '22+)

If $S_i(V) \cap S_j(V) = \varnothing$ for all distinct $i, j \in I$ then $\dim_{\mathrm{A}} F = \max\{\dim_{\mathrm{H}} F, \dim_{\mathrm{A}} P\}$.

Dimension interpolation

- Intermediate dimensions (Falconer–Fraser–Kempton, '20): for $\theta \in (0,1)$,

$$\dim_{\mathrm{H}} F \le \overline{\dim}_{\theta} F \le \overline{\dim}_{\mathrm{B}} F.$$

- Assouad spectrum (Fraser–Yu, '18): for $\theta \in (0,1)$,

$$\overline{\dim}_{\mathrm{B}} F \le \overline{\dim}_{\mathrm{A}}^{\theta} F \le \dim_{\mathrm{A}} F.$$

Hausdorff dimension

Alternative definition of Hausdorff dimension:

$$\dim_{\mathrm{H}} F = \inf\{\, s \geq 0 : \text{for all } \varepsilon > 0 \text{ there exists a finite or countable cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } \sum_i |U_i|^s \leq \varepsilon \,\}$$

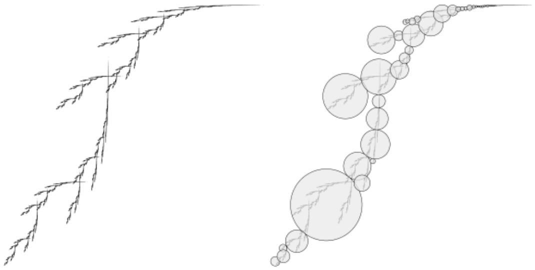

Figure: A cover using balls of different sizes. Picture by Jonathan Fraser.

Box dimension

Alternative definition of box dimension:

$$\overline{\dim}_{\mathrm{B}} F = \inf\{\, s \geq 0 : \text{for all } \varepsilon > 0 \text{ there exists } \delta_0 \in (0,1] \text{ such that for all } \delta \in (0, \delta_0) \text{ there exists a cover } \{U_1, U_2, \dots\} \text{ of } F \text{ such that } |U_i| = \delta \text{ for all } i, \text{ and } \sum_i |U_i|^s \leq \varepsilon \,\}.$$

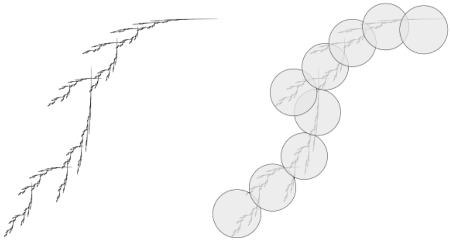

Figure: A cover using balls of the same size. Picture by Jonathan Fraser.

Intermediate dimensions

Upper θ-intermediate dimension for $\theta \in (0,1)$:

$$\overline{\dim}_\theta F = \inf\{\, s \geq 0 : \text{for all } \varepsilon > 0 \text{ there exists } \delta_0 \in (0,1] \text{ such that for all } \delta \in (0,\delta_0) \text{ there exists a cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } \delta^{1/\theta} \leq |U_i| \leq \delta \text{ for all } i, \text{ and } \sum_i |U_i|^s \leq \varepsilon \,\}.$$

Theorem (B.–Fraser, to appear in TAMS)

$\overline{\dim}_\theta F = \max\{\dim_{\mathrm{H}} F, \overline{\dim}_\theta P\}$.

Graph of the intermediate dimensions

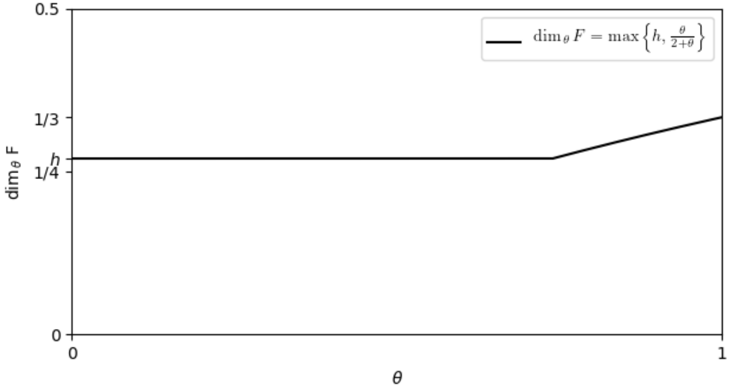

Figure: Intermediate dimensions when $P = \{ k^{-2} : k \in \mathbb{N} \}$

Assoud spectrum

The (upper) Assouad spectrum gives information about the thickest parts of the set with restriction on the relative scales according to θ:

$$\overline{\dim}_{\mathrm{A}}^{\theta} F := \inf \Big\{ \alpha : \text{ there exists } C > 0 \text{ such that for all } x \in F \text{ and } 0 < R < 1, r \leq R^{1/\theta}, \text{ we have } N_r(B(x,R) \cap F) \leq C(R/r)^{\alpha} \Big\}.$$

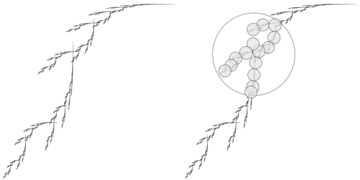

Figure: Covering a ball for the Assouad spectrum. Picture by Jonathan Fraser.

Assoud spectrum

Theorem (B.–Fraser, '22+)

$$\max\{\dim_{\mathrm{H}} F, \overline{\dim}_{\mathrm{A}}^{\theta} P\} \leq \overline{\dim}_{\mathrm{A}}^{\theta} F \leq \max_{\phi \in [\theta,1]} f(\theta,\phi),$$

where for $\theta \in (0,1)$ and $\phi \in [\theta, 1]$,

$$f(\theta,\phi) := \frac{(\phi^{-1}-1)\overline{\dim}_{\mathrm{A}}^{\phi} P + (\theta^{-1}-\phi^{-1})\overline{\dim}_{\mathrm{B}} F}{\theta^{-1}-1}.$$

In particular, $f(\theta,\theta) = \overline{\dim}_{\mathrm{A}}^{\theta} P$ and $f(\theta,1) = \overline{\dim}_{\mathrm{B}} F$.

Assouad spectrum - sharpness

If $P = \{ k^{-p} : k \in \mathbb{N} \}$ then Fraser and Yu ('18) proved that

$$\overline{\dim}_{\mathrm{A}}^{\theta} P = \min \left\{ \frac{1}{(1+p)(1-\theta)}, 1 \right\}$$

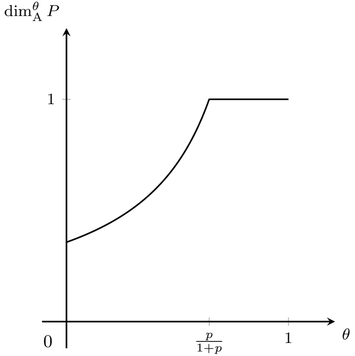

Figure: Case $p = 1.8$

Assoud spectrum - sharpness

Choosing the contraction ratios $c_k = k^{-t}$ for fixed $t \in [p+1, p+h^{-1}]$ and all large k shows that the bounds are sharp:

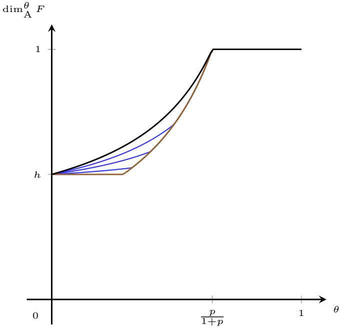

Figure: Graph of the Assouad spectrum for different values of t, with $p = 1.8$, $h = 0.5$

These are the first dynamically generated fractals with Assouad spectrum having two phase transitions.

Thank you for listening.

Happy Birthday Károly!