Interpolating between the Hausdorff and box dimensions

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Dimensions

- Recall that there are many notions of fractal dimension which can take different values for the same set.
- This means that how to define dimension is a non-trivial question.
- How/when/why do different dimensions of the same set differ?

Interpolating between dimensions

- If dim and Dim are notions of dimensions for which $\dim F \leq \mathrm{Dim} F$ for all 'reasonable' sets F, then we want to find a geometrically natural family of dimensions that lie between $\dim F$ and $\mathrm{Dim} F$ for all such sets.
- The family of dimensions should have some similarities with both dim and Dim, should satisfy properties that would be expected of all notions of dimension, and should lead to an interesting theory.
- This can lead to a better understanding of the similarities and differences between dim and Dim.
- Example: the *Assouad spectrum* (Fraser and Yu, 2018) and ϕ-Assouad dimensions (Fraser-Yu (2018), García-Hare-Mendivil (2019)) interpolate between the box and Assouad dimensions.

Hausdorff and box dimensions

- Roughly speaking, a disc has Hausdorff dimension 2 because it has positive and finite area, and it has box dimension 2 because the number of discs of size r needed to cover it scales approximately like r^{-2} as $r \to 0^+$.
- Throughout, $F \subset \mathbb{R}^d$ will be non-empty and bounded.
- $\dim_{\mathrm{H}} F \leq \overline{\dim}_{\mathrm{B}} F$.
- Definitions (equivalent to the usual definitions):
- $$\begin{aligned}\overline{\dim}_B F = \inf\{\, s \geq 0 : {} & \text{for all } \epsilon > 0 \text{ there exists } \delta_0 \in (0,1] \text{ such that for all} \\ & \delta \in (0,\delta_0) \text{ there exists a cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such} \\ & \text{that } |U_i| = \delta \text{ for all } i, \text{ and } \sum_i |U_i|^s \leq \epsilon \,\}.\end{aligned}$$
- $$\begin{aligned}\dim_H F = \inf\{\, s \geq 0 : {} & \text{for all } \epsilon > 0 \text{ there exists a finite or countable cover} \\ & \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } \sum_i |U_i|^s \leq \epsilon \,\}\end{aligned}$$

Intermediate dimensions - definition

- Falconer, Fraser and Kempton (2020) noted that these "may be regarded as two extreme cases of the same definition, one with no restriction on the size of covering sets, and the other requiring them all to have equal diameters" and defined the upper θ-intermediate dimension of F for $\theta \in (0,1)$ by

$$\overline{\dim}_\theta F = \inf\{\, s \geq 0 : \text{for all } \epsilon > 0 \text{ there exists } \delta_0 \in (0,1] \text{ such that for all } \delta \in (0,\delta_0) \text{ there exists a cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } \delta^{1/\theta} \leq |U_i| \leq \delta \text{ for all } i, \text{ and } \sum_i |U_i|^s \leq \epsilon \,\}.$$

- As expected, $\dim_{\mathrm{H}} F \leq \overline{\dim}_\theta F \leq \overline{\dim}_{\mathrm{B}} F$ for all $\theta \in (0,1)$.
- There is also a lower version of the dimensions.

Properties

- Monotonicity: the function $\theta \mapsto \overline{\dim}_\theta F$ is increasing in $\theta \in [0,1]$.
- Continuity: FFK showed that $\theta \mapsto \overline{\dim}_\theta F$ is continuous for $\theta \in (0,1]$. More than this, they gave a quantitative continuity result which can be improved to the following:

Proposition (B., 2020)

If $0 < \theta \le \phi \le 1$ then

$$\overline{\dim}_\theta F \le \overline{\dim}_\phi F \le \overline{\dim}_\theta F + \frac{(\phi-\theta)\overline{\dim}_\theta F(\dim_{\mathrm{A}} F - \overline{\dim}_\theta F)}{(\phi-\theta)\overline{\dim}_\theta F + \theta \dim_{\mathrm{A}} F}.$$

- Of particular interest is the case $\phi = 1$, which upon rearranging gives the general lower bound

$$\overline{\dim}_\theta F \ge \frac{\theta \dim_{\mathrm{A}} F \overline{\dim}_{\mathrm{B}} F}{\dim_{\mathrm{A}} F - (1-\theta)\overline{\dim}_{\mathrm{B}} F}.$$

- For some interesting fractals, for example some Bedford-McMullen carpets, this gives the best known lower bound for some θ.

Examples - polynomial sequences

- For $p \in (0, \infty)$ define $F_p := \{0\} \cup \{\, n^{-p} : n \in \mathbb{N} \,\}$.
- FFK showed that $\dim_\theta F_p = \frac{\theta}{p+\theta}$ for all $\theta \in [0, 1]$.

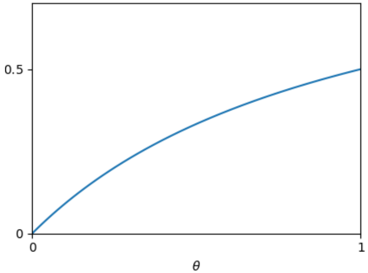

Figure: Intermediate dimensions of $\{\, 1/b : b \in \mathbb{N} \,\}$

- These examples show that my quantitative continuity result and general lower bound above are sharp.

Discontinuity at $\theta = 0$

- Define $F_{log} := \{0\} \cup \{\, 1/(\log n) : n \in \mathbb{N}\,\}$.
- Straightforward to show $\dim_{\mathrm{B}} F_{log} = 1$.
- Since 1 is also the ambient spatial dimension, by a result of FFK, $\dim_\theta F_{log} = 1$ for all $\theta \in (0, 1]$.
- But F_{log} is countable so has Hausdorff dimension 0
- In particular, the intermediate dimensions of F_{log} are discontinuous at $\theta = 0$ and do not interpolate all the way between the Hausdorff and box dimensions.
- In fact there are many compact sets whose intermediate dimensions are discontinuous at $\theta = 0$.

Recovering the interpolation

Theorem (B., 2020)

If $F \subset \mathbb{R}^d$ is non-empty and compact then for all $s \in [\dim_{\mathrm{H}} F, \overline{\dim}_{\mathrm{B}} F]$ there exists a function $\Phi_s : (0,1) \to (0,1)$ that is increasing and satisfies $\Phi_s(\delta)/\delta \to 0$ as $\delta \to 0^+$ such that if we define the new dimension

$$\overline{\dim}^{\Phi_s} F = \inf\{\, s \geq 0 : \text{for all } \epsilon > 0 \text{ there exists } \delta_0 \in (0,1] \text{ such that for all } \delta \in (0, \delta_0) \text{ there exists a cover } \{U_1, U_2, \dots\} \text{ of } F \text{ such that } \Phi_s(\delta) \leq |U_i| \leq \delta \text{ for all } i, \text{ and } \sum_i |U_i|^s \leq \epsilon \,\}$$

then $\overline{\dim}^{\Phi_s} F = s$.

- It can be shown that $\overline{\dim}^{\Phi_s}$ is a 'reasonable' notion of dimension with many of the same properties as $\overline{\dim}_\theta$, and these dimensions will distinguish between sets which the usual notions of dimension will not distinguish between.
- The compact assumption cannot be removed in general: consider $\mathbb{Q} \cap [0,1]$.
- The choice of Φ_s is not unique, and depends on F.

References

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Thank you for listening!

Questions?