# Orthogonal projections of fractals and the digital sundial

Amlan Banaji[1]

University of St Andrews

# Orthogonal projections

We are interested in the orthogonal projection of a subset $F \subseteq \mathbb{R}^d$ onto a lower-dimensional subspace. We start by considering $\pi \colon \mathbb{R}^2 \to \mathbb{R}$ for simplicity.

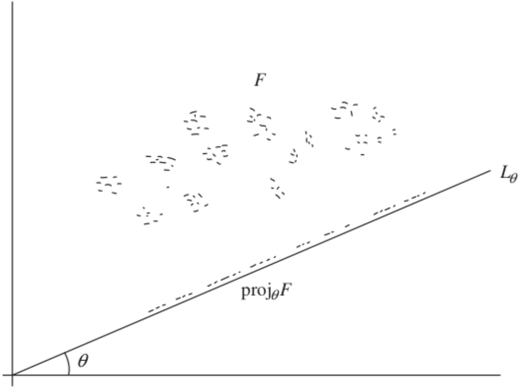


Figure: Picture by Kenneth Falconer

# Dimensions of projections

- A 0-dimensional point in $\mathbb{R}^2$ is projected onto a $d$-dimensional point in $\mathbb{R}$. A 1-dimensional curve in $\mathbb{R}^2$ is usually projected onto a 1-dimensional set with positive length in $\mathbb{R}$, but not always.
- One might expect the following to hold in many situations:

$$\dim \pi(F) = \min\{\dim F, 1\}$$

- If $F$ is fractal, we use fractal dimension.

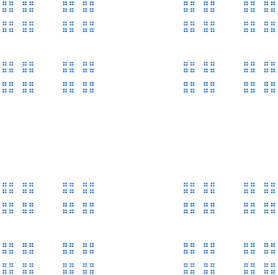

Figure: Cantor dust (based on image at hyperlink by WalkingRadiance, CC-BY-SA-4.0)

# Hausdorff measure

For $s \geq 0$ and $\delta > 0$, define the Hausdorff content

$$H^s_\delta(F) = \inf\left\{ \sum_{i=1}^{\infty} (\mathrm{diam}(U_i))^s \,\middle|\, F \subseteq \bigcup_i U_i, \mathrm{diam}(U_i) \leq \delta \right\}.$$

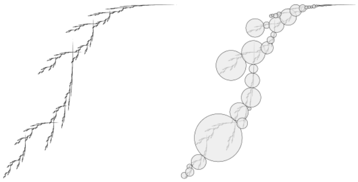

Figure: A cover using balls of different sizes, by Jonathan Fraser

As $\delta$ decreases, the content converges to a limit called the $s$-dimensional Hausdorff measure of $F$:

$$H^s_\delta(F) \to H^s(F) \in [0,\infty] \text{ as } \delta \to 0^+.$$

# Hausdorff dimension

- For each $F$ there is a unique $s \geq 0$, called the Hausdorff dimension of $F$, such that if $0 \leq t < s$ then $H^t(F) = \infty$ and if $t > s$ then $H^t(F) = 0$.

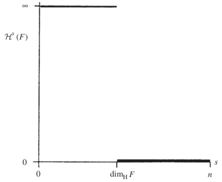


Figure: Graph of the $s$-dimensional Hausdorff measure of a set against $s$, by Kenneth Falconer

- Intuitively, line segment has Hausdorff dimension 1 because it has positive and finite length.
- Simpler definition:

$$\dim_{\mathrm{H}} F = \inf\{\, s \geq 0 : \text{for all } \varepsilon > 0 \text{ there exists a finite or countable cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } \sum_i |U_i|^s \leq \varepsilon \,\}.$$

# Marstrand's projection theorem

**Theorem (Marstrand, 1964)**

*If $F \subseteq \mathbb{R}^2$ be Borel then for almost all $\theta \in [0, \pi)$,*

$$\dim_{\mathrm{H}} \pi_\theta F = \min\{\dim_{\mathrm{H}} F, 1\}.$$

If $\dim_{\mathrm{H}} F > 1$ then $\pi_\theta F$ has positive Lebesgue measure (length) in $L_\theta$ for almost all $\theta$.

# Proof (Kaufman, 1968)

Upper bound: $\dim_{\mathrm{H}} \pi_\theta F \leq 1$ since $\pi_\theta F$ is contained in a line, and $\dim_{\mathrm{H}} \pi_\theta F \leq \dim_{\mathrm{H}} F$ since $\pi_\theta$ cannot increase distances.
Lower bound: use the potential theoretic characterisation of Hausdorff dimension, essentially due to Frostman (1935):

$$\dim_{\mathrm{H}} F = \sup \left\{ s \geq 0 : \text{there exists a probability measure } \mu \text{ on } F \text{ with} \right. \\ \left. \int_F \int_F \frac{d\mu(x) d\mu(y)}{||x-y||^s} < \infty \right\}$$

# Proof (lower bound continued)

Given $s < \min\{\dim_{\mathrm{H}} F, 1\}$ and a measure $\mu$ on $F$, with that double integral finite, project it to a measure $\mu_\theta$ on $\pi_\theta F$. Then

$$\begin{aligned}\int_0^\pi \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{d\mu_\theta(u) d\mu_\theta(v)}{|u-v|^s} d\theta &= \int_0^\pi \int_F \int_F \frac{d\mu(x) d\mu(y)}{|x \cdot \underline{\theta} - y \cdot \underline{\theta}|^s} d\theta \\ &= \int_F \int_F \int_0^\pi \frac{d\theta}{|\underline{\tau_{x-y}} \cdot \underline{\theta}|^s} \frac{d\mu(x) d\mu(y)}{||x-y||^s} \\ &\leq C_s \int_F \int_F \frac{d\mu(x) d\mu(y)}{||x-y||^s} < \infty.\end{aligned}$$

Hence for almost every $\theta \in [0, \pi)$,

$$\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{d\mu_\theta(u) d\mu_\theta(v)}{|u-v|^s} < \infty$$

and $\dim_{\mathrm{H}} \pi_\theta F \geq s$. $\square$

# Box dimension

- (Upper) box dimension is defined by

$$\overline{\dim}_{\mathrm{B}} F := \limsup_{\delta \to 0^+} \frac{\log N_\delta(F)}{-\log \delta},$$

where $N_\delta(F)$ is the smallest number of balls of radius $\delta$ needed to cover $F$.

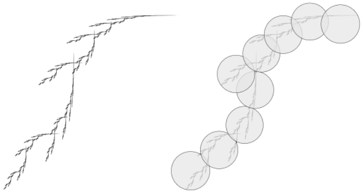

Figure: A cover using balls of the same size, by Jonathan Fraser

- Intuitively, a line segment has box dimension 1 because the number of discs of size $\delta$ needed to cover it scales approximately like $\delta^{-1}$ as $\delta \to 0^+$.
- For almost every $\theta$, $\overline{\dim}_{\mathrm{B}} \pi_\theta F$ takes a constant value depending only on $F$, but surprisingly this value may be smaller than $\min\{\overline{\dim}_{\mathrm{B}} F, 1\}$.

# Exceptional directions

- Can we bound the size of the set of $\theta$ for which the dimension of the projection is smaller than expected?
- Kaufman (1968) proved that

$$\dim_{\mathrm{H}}\{\,\theta \in [0,\pi) : \dim_{\mathrm{H}} \pi_\theta F < \dim_{\mathrm{H}} F\,\} \le \dim_{\mathrm{H}} F.$$

- Obtaining sharp estimates for exceptional directions for Hausdorff and box dimension is an area of active research.
- If $F$ has more structure then we expect the exceptional set to be very small. If $S_1,\ldots,S_k\colon\mathbb{R}^2\to\mathbb{R}^2$ are similarities ($||S_i(x)-S_i(y)||=c_i||x-y||$) then Hutchinson (1981) showed that there is a unique non-empty compact self-similar set satisfying

$$F=\bigcup_{i=1}^{k} S_i(F).$$

  In this case, Wu (2022, unpublished) has proved that $\{\,\theta\in[0,\pi) : \dim_{\mathrm{H}}\pi_\theta F < \min\{\dim_{\mathrm{H}} F, 1\}\,\}$ is at most countable.


Figure: Shadows, by Ian Stewart

# Digital sundial

What can be said about the set of shadows that a given object can make?
If the dimension of the set equals the dimension of the subspace, the dimension of almost every projection is the same, but other than this almost anything can happen!

**Theorem (Falconer, 1987)**

*Let $G_x$ be any subset of the plane through the origin in direction $x$, for each $x \in \mathbb{RP}^2$ such that $\bigcup_{x \in \mathbb{RP}^2} G_x$ is measurable.*
*Then there exists a Borel set $F \subset \mathbb{R}^3$ of Hausdorff dimension 2 such that for almost all $x \in \mathbb{RP}^2$ the 2-dimensional Hausdorff measure of the symmetric difference between $\pi_x(F)$ and $G_x$ is 0.*

# Digital sundial

In particular, there exists a digital sundial! But those that have been manufactured work by a different principle.

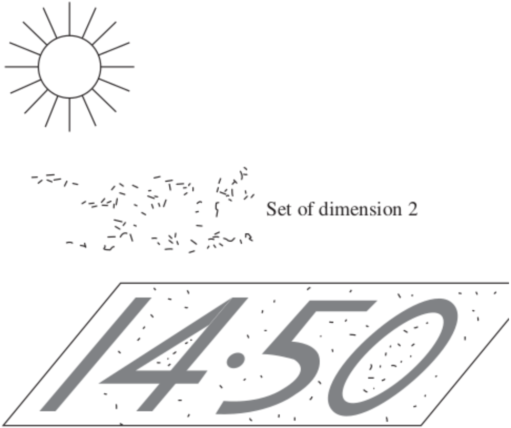


Figure: A digital sundial, by Kenneth Falconer

# Digital sundial

The proof uses an ‘iterated Venetian blind’ construction.

Figure: Venetian Blinds by Bob Jenkins at hyperlink, CC BY 2.0

# Digital sundial

The proof uses an ‘iterated Venetian blind’ construction.

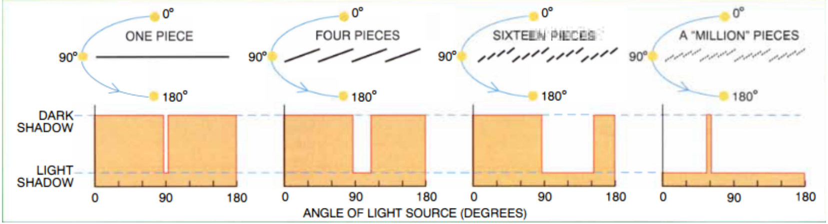


Figure: Picture by Ian Stewart

Thank you for listening!

Questions welcome