

Below is a standalone proof of the classical Central Limit Theorem based on the proof in the Geiss–Geiss book.

Theorem 1 (classical CLT). *Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and let $f_1, f_2, \dots, \Omega \rightarrow \mathbb{R}$ be a sequence of i.i.d. random variables such that $\mathbb{E}(f_n^2) = 1$ and $\mathbb{E}f_n = 0$. If $S_n := f_1 + \dots + f_n$, then one has*

$$\frac{S_n}{\sqrt{n}} \xrightarrow{d} g \quad \text{with} \quad \text{law}(g) = \mathcal{N}_{0,1}.$$

For the proof we will need the following lemma.

Lemma 2 (complex number estimate). *For all $n \in \mathbb{N}$ and $z, v \in \mathbb{C}$ it holds that*

$$|z^n - v^n| \leq |z - v| \cdot n \cdot \max(|z|, |v|)^{n-1}.$$

Proof. We use the factorisation

$$z^n - v^n = (z - v)(z^{n-1} + z^{n-2}v + \dots + zv^{n-2} + v^{n-1}).$$

The triangle inequality now gives

$$|z^n - v^n| \leq |z - v| \sum_{k=0}^{n-1} |z|^{n-1-k} |v|^k \leq |z - v| \cdot n \cdot \max(|z|, |v|)^{n-1},$$

as required. □

We can now prove Theorem 1.

Proof of Theorem 1. We have to show that

$$\mathbb{P}\left(\frac{1}{\sqrt{n}}(f_1 + \dots + f_n) \leq x\right) \xrightarrow{n \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy$$

for all $x \in \mathbb{R}$, or, putting $S_n := f_1 + \dots + f_n$, that $\frac{1}{\sqrt{n}}S_n \Rightarrow g \sim \mathcal{N}_{0,1}$. By the Portmanteau theorem (Theorem 12.1.1) this is equivalent to

$$\varphi_{\frac{1}{\sqrt{n}}S_n}(t) \xrightarrow{n \rightarrow \infty} \varphi_g(t) = e^{-\frac{t^2}{2}}$$

for all $t \in \mathbb{R}$. Now by independence and Theorem 11.9.1, and then Sheet 6 Problem 4,

$$\varphi_{\frac{1}{\sqrt{n}}S_n}(t) = \varphi_{\frac{1}{\sqrt{n}}(f_1 + \dots + f_n)}(t) = \varphi_{\frac{f_1}{\sqrt{n}}}(t) \cdots \varphi_{\frac{f_n}{\sqrt{n}}}(t) = \varphi\left(\frac{t}{\sqrt{n}}\right)^n,$$

where $\varphi(t) = \varphi_{f_1}(t)$. Since $\mathbb{E}(f_1^2) < \infty$, Proposition 11.6.1 implies that φ'' exists and is continuous and bounded with $\varphi''(0) = i^2 \mathbb{E}(f_1^2) = -\mathbb{E}(f_1^2)$ and $\varphi'(0) = i \mathbb{E}f_1$. By Taylor's theorem we have

$$\begin{aligned} \varphi(t) &= \varphi(0) + t\varphi'(0) + \frac{t^2}{2}\varphi''(0) + o(t^2) \\ &= \varphi(0) + ti\mathbb{E}f_1 + \frac{t^2}{2}i^2\mathbb{E}(f_1^2) + o(t^2) \\ &= 1 - \frac{t^2}{2} + o(t^2) \end{aligned}$$

for $t \in \mathbb{R}$ with

$$\lim_{t \rightarrow 0} \frac{o(t^2)}{t^2} = 0.$$

So it remains to show that

$$\varphi_{\frac{1}{\sqrt{n}}} S_n(t) = \left(1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right)^n \xrightarrow{n \rightarrow \infty} e^{-\frac{t^2}{2}}.$$

Properties of the exponential function tell us that $\left(1 - \frac{t^2}{2n}\right)^n \rightarrow e^{-\frac{t^2}{2}}$ as $n \rightarrow \infty$. Therefore it suffices to show that

$$\left| \left(1 - \frac{t^2}{2n}\right)^n - \varphi_{\frac{1}{\sqrt{n}}} S_n(t) \right| = \left| \left(1 - \frac{t^2}{2n}\right)^n - \left(1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right)^n \right| \xrightarrow{n \rightarrow \infty} 0.$$

We use Lemma 2 setting $z := 1 - \frac{t^2}{2n}$ and $v := 1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)$, where we note that $|z|, |v| \leq 1$, which implies

$$\left| \left(1 - \frac{t^2}{2n}\right)^n - \left(1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right)^n \right| \leq n \left| o\left(\frac{t^2}{n}\right) \right| \xrightarrow{n \rightarrow \infty} 0. \quad \square$$