Fourier decay for nonlinear pushforwards of self-similar measures

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Fourier transform of measures

- **Fourier transform** of a measure μ on $\mathbb{R}^k$ is the function $\widehat{\mu}\colon \mathbb{R}^k \to \mathbb{C}$,
$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^k} e^{-2\pi i \langle \xi, x \rangle} d\mu(x).$$
- Decay rates of $|\widehat{\mu}(\xi)| \to 0$ as $|\xi| \to \infty$ have applications to normality of μ-typical points, Fourier uniqueness, Fourier restriction...
- If μ is Cantor–Lebesgue measure,
$$\widehat{\mu}(1) = \widehat{\mu}(3) = \widehat{\mu}(9) = \cdots$$
but $|\widehat{\mu}(\xi)|$ decays outside a "zero-dimensional" set of ξ.

Nonlinear pushforwards

Principle: **nonlinear** dynamically defined measures often have polynomial Fourier decay.

Theorem (Kaufman, 1984)

If μ is Cantor–Lebesgue and $f(x) = x^2$ then the pushforward satisfies

$$|\widehat{f_*\mu}(\xi)| \lesssim |\xi|^{-\eta} \qquad \text{for some } \eta > 0.$$

Can η be **quantified**?
$\eta = 0.01$ works (Mosquera–Shmerkin, 2018)
$\eta = 0.06$ works (B.–Yu, 2024+)

Open question

Does arbitrary $\eta < \frac{1}{2}\frac{\log 2}{\log 3} \approx 0.32$ work (in which case $\mathrm{supp}(f_*\mu)$ is a Salem set)?

Self-similar measures

Fix $r_1, \ldots, r_m \in (0,1)$, **commuting** $k \times k$ orthogonal maps $O_1, \ldots, O_m$, vectors $t_1, \ldots, t_m \in \mathbb{R}^k$, and weights $p_1, \ldots, p_m \in (0,1)$ with $p_1 + \cdots + p_m = 1$. The **self-similar measure** μ satisfies

$$\mu(A) = \sum_{i \in I} p_i \mu(S_i^{-1}(A)).$$

Notation:

- A: Assouad dim of $\mathrm{supp}(\mu)$
- F: Frostman exponent of μ
- $\kappa_p \coloneqq \sup \left\{ s \geq 0 : \int_{B(0,R)} |\widehat{\mu}(\xi)|^p d\xi \lesssim R^{k-s} \right\}$: Fourier ℓ^p dimension of μ.

Always $\kappa_1 \leq F \leq \kappa_2 \leq A$.
If OSC & measure of max dim: $\kappa_1 \leq F = \kappa_2 = A$.

Quantitative decay

Let μ be as above, and $f\colon \mathbb{R}^k \to \mathbb{R}$ is C^2 have graph $\{x, f(x)\}_{x\in\mathbb{R}^k} \subset \mathbb{R}^{k+1}$ having **non-vanishing Gaussian curvature** over $\text{supp}(\mu)$.

Theorem (B.–Yu)

If $F > k/2$ then $|\widehat{f_*\mu}(\xi)| \lesssim |\xi|^{-\eta}$ for $\eta > 0$ indep. of f. Can take any

$$0 < \eta < \max\left\{\frac{2\kappa_2 - k}{4 + 2A - k}, \frac{F + \kappa_1 - k}{2 - k + 2A + \kappa_1 - F}\right\}.$$

Missing digit measures

- Let μ_b be the natural self-similar measure on the set of numbers whose base-b expansion misses digit 0.
- Chow–Varjú–Yu (2024+): $\kappa_1 \to 1$ as $b \to \infty$.

Corollary

If $f : \mathbb{R} \to \mathbb{R}$ has $f'' > 0$, then $|\widehat{f_* \mu_b}(\xi)| \lesssim |\xi|^{-\eta}$, where we can take $\eta = 1/3 - o_b(1)$.

If μ_b were Salem (this is not known) then one could take $\eta = 1/2 - o_b(1)$.

Nonlinear arithmetic

- Arithmetic product of $X, Y \subseteq \mathbb{R}$ is $X \cdot Y := \{xy : x \in X, y \in Y\} \subseteq \mathbb{R}$.
- Multiplicative convolution $\mu \cdot \nu$ of measures on $\mathbb{R}$ is pushforward of $\mu \times \nu$ under $f(x, y) = xy$.
- If X, Y are "large" (dimension), when does this imply $X \cdot Y$ is "large" (positive Lebesgue measure)?
- Generalised Marstrand projection theorem (Peres–Schlag, 2000): $K \subset \mathbb{R}^2$ compact, $\dim_{\mathrm{H}} K > 1$, P_a a family of smooth maps (satisfying conditions), then $P_a(K)$ has positive measure for "almost every" a.

Arithmetic of self-similar sets

When can exceptional directions be removed?

Theorem (Hochman–Shmerkin, 2012)

If self-similar set E has some contraction ratio s and F has a contraction ratio t with $\log s / \log t \notin \mathbb{Q}$ then $\dim_{\mathrm{H}}(E \cdot F) = \min\{\dim_{\mathrm{H}} E + \dim_{\mathrm{H}} F, 1\}$.

Conjectures

- Let E, F be self-similar sets in $\mathbb{R}$. If $\dim_{\mathrm{H}} E + \dim_{\mathrm{H}} F > 1$, then $\mathrm{Leb}(E \cdot F) > 0$.
- Let μ, ν be self-similar measures on $\mathbb{R}$. If $\dim_{\mathrm{H}} \mu + \dim_{\mathrm{H}} \nu > 1$ then $\mu \cdot \nu$ is absolutely continuous.

Note $f(x, y)$ is a **nonlinear** projection! Won't work for linear projections or $E + F$ (recall Ian Morris's talk).

Progress towards conjecture for measures

Theorem (B.–Yu)

- If μ is self-similar with SSC and natural weights and

$$\dim_{\mathrm{H}} \mu > (\sqrt{65} - 5)/4 \approx 0.765...$$

  then $\mu \cdot \mu$ has an L^2 density.
- If we only assume exponential separation (and consider the measure of maximal dimension), conclusion holds when $\dim_{\mathrm{H}} \mu > 7/9 \approx 0.777...$.

Progress for sets

No separation conditions!

Theorem (B.–Yu)

If $E, F \subset \mathbb{R}$ are self-similar with $\min\{\dim_{\mathrm{H}} E, \dim_{\mathrm{H}} F\} > (\sqrt{65} - 5)/4$ then $\mathrm{Leb}(E \cdot F) > 0$.
Also if $R_{(a,b)}$ is the radial projection from (a, b) then $\mathrm{Leb}(R_{a,b}(F \times F)) > 0$ for **all** $(a, b) \notin F \times F$.

Proof idea: take a SSC subsystem with ε dimension loss, put natural measure on it, apply corollary for measures.

Thank you for listening!