# Fourier decay of non-linear images of self-similar measures

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[1]Based on joint work with Simon Baker https://arxiv.org/abs/2401.01241

# Fourier transform

- The Fourier transform $\hat{\mu}: \mathbb{R} \to \mathbb{C}$ of a Borel probability measure $\mu$ on $\mathbb{R}$ is

$$\hat{\mu}(\xi) := \int_{-\infty}^{\infty} e(\xi x) d\mu(x),$$

  where $e(t) := e^{-2\pi i t}$.

- Given a measure, one can ask:
  - Is $\mu$ Rajchman? Does $\hat{\mu}(\xi) \to 0$ as $|\xi| \to \infty$?
  - If so, does it have polynomial Fourier decay

$$|\hat{\mu}(\xi)| \leq C|\xi|^{-\varepsilon} \qquad \text{for all } \xi \neq 0?$$

# Self-similar measures

- Given an iterated function system (IFS) $\{\phi_i(x) = r_i x + t_i\}_{i=1}^{l}$, and positive weights $p_1 + \cdots + p_m = 1$, the self-similar measure $\mu$ satisfies

$$\mu(B) = \sum_{i=1}^{l} p_i \mu(\phi_i^{-1}(B)) \qquad \text{for all Borel } B \subset \mathbb{R}.$$

  Throughout, we assume that $\mu$ is not a single atom.
- Solomyak ('21): outside set of exceptional parameters with zero Hausdorff dimension, self-similar measures have polynomial Fourier decay.
- The $(1/2, 1/2)$ measure on the middle-third Cantor set is not Rajchman.

# Results

Let $\mu$ be any non-atomic self-similar measure on $\mathbb{R}$ (we impose no homogeneity or separation assumptions).

## Theorem (Baker–B. / Algom–Chang–Wu–Wu)

Let $F\colon \mathbb{R} \to \mathbb{R}$ be $C^2$ with $F''(x) > 0$ for all $x \in \mathbb{R}$. There exists $\varepsilon = \varepsilon(\mu) > 0$ (independent of $F$) and $C = C(\mu, F) > 0$ such that

$$|\widehat{F\mu}(\xi)| \leq C|\xi|^{-\varepsilon} \quad \text{for all } \xi \neq 0.$$

Combining with Davenport–Erdős–LeVeque ('63) gives:

## Corollary

If $F\colon \mathbb{R} \to \mathbb{R}$ is

- $C^2$ with $F'' > 0$, or
- a polynomial with $\deg(f) \geq 2$,

then $F\mu$-almost every number is normal in every base.

# Disintegrating $\mu$

- The theorem is already known when $\mu$ is homogeneous (Kaufman ('84), Chang–Gao ('17+), Mosquera–Shmerkin ('18)), since then $\mu$ is an infinite convolution so $\hat{\mu}$ is an infinite product.
- Iterate our inhomogeneous IFS to get two well-separated maps $\phi_1, \phi_2$ with the same contraction ratio and probability.
  Given words $\mathbf{a}, \mathbf{b}$ of fixed length $k \gg 1$, write $\mathbf{a} \sim \mathbf{b}$ if $\forall i$ either $a_i = b_i$ or $a_i, b_i \in \{1, 2\}$. Put a probability measure $P$ on the space $\Omega$ of infinite sequences of equivalence classes. Given $\omega \in \Omega$, define a *statistically* self-similar measure $\mu_\omega$ supported on $\{\, \pi(\omega') : \forall n, \omega'_n \in [\omega_n] \,\}$. Then

$$\mu = \int_\Omega \mu_\omega \, dP(\omega).$$

- Each $\mu_\omega$ is an infinite convolution

$$\mu_\omega = *_{m=1}^{\infty} \frac{1}{\#[\mathbf{a}_m]} \sum_{i \in \Delta_{[\mathbf{a}_m]}} \delta_{t_i \cdot \prod_{j=1}^{m-1} r_{[\mathbf{a}_j]}}.$$

  Therefore

$$\widehat{\mu_\omega}(\xi) = \prod_{m=1}^{\infty} \frac{1}{\#[\mathbf{a}_m]} \sum_{i \in \Delta_{[\mathbf{a}_m]}} e\Big(\xi \cdot t_i \cdot \prod_{j=1}^{m-1} r_{[\mathbf{a}_j]}\Big).$$

# Decay outside sparse frequencies

- Kaufman ('84), Tsujii ('15): $|\hat{\mu}(\xi)| \leq |\xi|^{-\delta}$ outside a 'sparse' set of frequencies $\xi$.
- We use large deviation theory and an Erdős–Kahane argument to find, for each $T' > 0$, $\Omega_{T'}$ with $P(\Omega \setminus \Omega_2) \leq C'_k (T')^{-\varepsilon_1}$ and such that $\forall T \geq T' \, \forall \omega \in \Omega_{T'}$,

$$\{ \xi \in [-T, T] : |\widehat{\mu_\omega}(\xi)| \geq T^{-\varepsilon_2} \}$$

can be covered by $C_k T^{o_k(1)}$ intervals of length 1.

# Non-linear images

- Writing $\mu_\omega = \mu_{N_\omega} * \lambda_{N_\omega}$ for a well-chosen $N_\omega$, and Taylor expanding $F$,

$$|\widehat{F\mu_\omega}(\xi)| = \Big| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e(\xi F(x+y)) d\mu_{N_\omega}(x) d\lambda_{N_\omega}(y) \Big|$$
$$\leq \int_{-\infty}^{\infty} \left| \widehat{\mu_{\sigma^{N_\omega}\omega}} \left( \xi F'(x) \prod_{n=1}^{N_\omega} r_{[\mathtt{a}_n]} \right) \right| d\mu_{N_\omega}(x) + C\xi^{-1/3}.$$

- Use $F'' > 0$ to bound (with high $P$-probability) the intervals of $\xi$ which do not result in decay for $\widehat{\mu_{\sigma^{N_\omega}\omega}}$.
  Bound the measure of these intervals by using large deviations theory to show that $\mu_{\sigma^{N_\omega}\omega}$ has Frostman exponent $s > 0$ (independent of $\omega$, $k$):

$$\mu_\omega((x, x+r)) \leq r^s.$$

- Conclusion of proof: writing $\Omega = \mathit{Good} \sqcup \mathit{Bad}$,

$$|\widehat{F\mu}(\xi)| \leq \int_{\mathit{Good}} |\widehat{F\mu_\omega}(\xi)| \, dP(\omega) + P(\mathit{Bad}) \leq C_1|\xi|^{-\delta} + C_2|\xi|^{-\eta}.$$

# Thank you for listening!

## Questions welcome