Metric spaces where geodesics are never unique

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[1] Based on https://arxiv.org/abs/2209.00598

Geodesics

- Geodesics describe the 'shortest paths' through a space.
- Throughout, (X, d) denotes a metric space with more than one point.

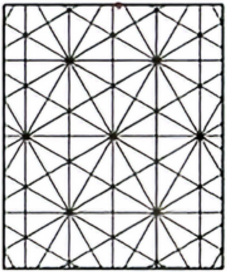
Definition

If $u, v \in X$ distinct, a *geodesic* from u to v is a function $\gamma \colon [0,1] \to X$ such that $\gamma(0) = u$, $\gamma(1) = v$, and

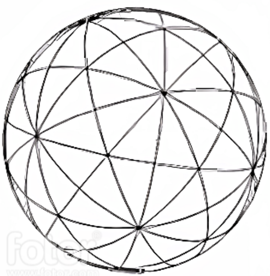
$$d(\gamma(s), \gamma(t)) = |s - t| d(u, v) \qquad \text{for all } s, t \in [0,1].$$

- They are important in general relativity.

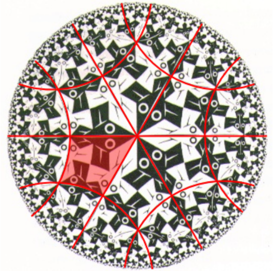
Uniqueness of geodesics



Euclidean
Uniquely geodesic
Based on image by Vladimir Marković



Spherical
Not uniquely geodesic
Based on image by Vladimir Marković



Hyperbolic (artwork by Escher)[2]
Uniquely geodesic

Definition

A metric space (X, d) is *multigeodesic* if for all distinct $u, v \in X$ there exist at least two distinct geodesics from u to v.

[2] Image by Anneke Bart and Bryan Clair
https://mathstat.slu.edu/escher/index.php/File:Hyp-circle-limit-i-tess.png

Normed vector spaces

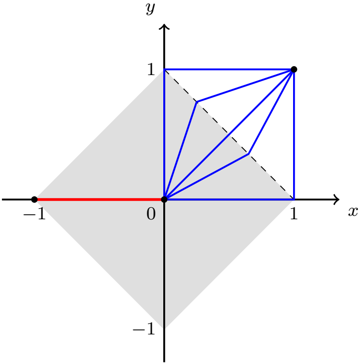


Figure: Geodesics in $(\mathbb{R}^2, ||\cdot||_1)$ where $||(x,y)||_1 := |x| + |y|$.

Theorem (B., to appear in *Amer. Math. Monthly*)

A real normed vector space $(X, ||\cdot||)$ is multigeodesic if and only if for all $x \in X$ with $||x|| = 1$ there exist $C \in (0,1)$ and $y \in X \setminus \{Cx\}$ such that $||y|| = C$ and $||x - y|| = 1 - C$.

L^1 spaces

Corollary

The space $(C([0,1]), ||\cdot||_1)$*, where* $||f||_1 := \int_0^1 |f(x)|dx$*, is multigeodesic.*

Proof.

Let $g\colon [0,1] \to \mathbb{R}$ be continuous with $||g||_1 = 1$. Define $h(x) = xg(x)$ and let $C := ||h||_1$.

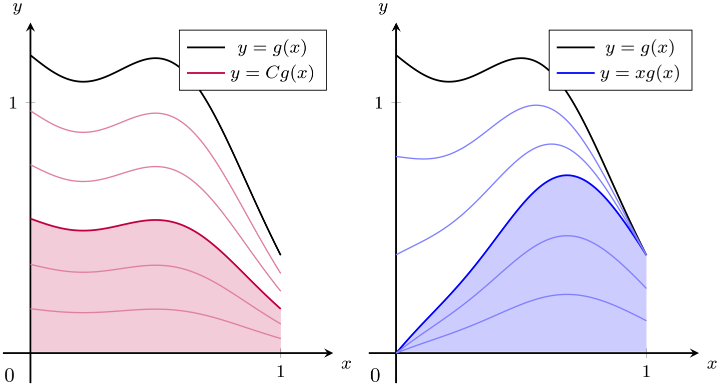
$$\begin{aligned} ||h||_1 + ||g-h||_1 &= \int_0^1 |xg(x)|dx + \int_0^1 |g(x) - xg(x)|dx \\ &= \int_0^1 (x + (1-x))|g(x)|dx \\ &= ||g||_1, \end{aligned}$$

so $||g-h||_1 = ||g||_1 - ||h||_1 = 1 - C$. Therefore by the theorem, $(C([0,1]), ||\cdot||_1)$ is multigeodesic. $\square$

L^1 spaces

Corollary

The space $(C([0,1]), ||\cdot||_1)$, where $||f||_1 := \int_0^1 |f(x)|dx$, is multigeodesic.



However, L^p spaces for $p > 1$ are not multigeodesic.

Laakso spaces

In general multigeodesic metric spaces, it might not be possible to make geodesics disjoint.

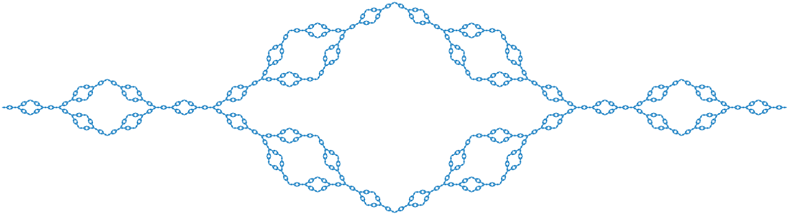


Figure: The Laakso space is multigeodesic.

Conjecture

No multigeodesic space can be embedded into $\mathbb{R}^n$ with a bi-Lipschitz map.

Thank you for listening!

Questions welcome