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Microthesis: Intermediate Dimensions

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Hausdorff and box dimension are two notions of fractal dimension, and the intermediate dimensions are a family of dimensions which lie between them. In this microthesis, we describe the form that these dimensions can take for general sets and for a class of self-affine fractal sets.

Fractal geometry

Much of the classical study of geometry relates to smooth objects such as manifolds, which have a well-defined integer dimension. However, many natural phenomena, such as the British coastline, display much more detailed and intricate structure across a range of scales, and often have some form of *self-similarity*, meaning that small parts of it have similar properties to the whole. Objects with these properties are often called *fractals*, though the term has no precise mathematical definition. The 'length' of the British coastline increases as we decrease the length δ of the ruler used to measure it, and scales very roughly like a constant multiple of $\delta^{-0.2}$ (i.e. we need to use $\approx \delta^{-1.2}$ rulers of length δ) over a range of scales. It therefore makes sense to regard the 'dimension' of the coastline as being 1.2.

To describe the theory of fractal dimensions more formally, we work with non-empty bounded subsets of Euclidean space $\mathbb{R}^d$ throughout. The box dimension of such a set F is defined by

$$\dim_{\mathrm{B}} F := \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

(if the limit exists), where $N_\delta(F)$ is the smallest number of open balls of diameter δ needed to cover F. There is another notion of dimension, called *Hausdorff dimension*, which is a lower bound for box dimension, and is perhaps more widely used across mathematics. It is defined by

$$\dim_{\mathrm{H}} F := \inf\{s \geq 0 : \text{for all } \varepsilon > 0 \text{ there is a cover } \{U_i\}_{i=1}^{\infty} \text{ of } F \text{ such that } \sum_i (\operatorname{diam} U_i)^s \leq \varepsilon\}.$$

Intermediate dimensions

The key difference between the two dimensions discussed above is that all the sets in a cover for box dimension have equal diameter δ, whereas for Hausdorff dimension the diameters of covering sets lie in an interval $[0,\delta]$. The intermediate dimensions $\dim_\theta F$ are defined by restricting the allowable diameters of sets in the cover to intervals of the form $[\delta^{1/\theta},\delta]$, where $\theta \in (0,1]$ is a fixed parameter. They satisfy $\dim_{\mathrm{H}} F \leq \dim_\theta F \leq \dim_{\mathrm{B}} F$ with $\dim_1 F = \dim_{\mathrm{B}} F$, and we define $\dim_0 F = \dim_{\mathrm{H}} F$. These dimensions provide more nuanced geometric information than Hausdorff or box dimension in isolation.

Definition 1. (Falconer–Fraser–Kempton [3]). For a set $F \subset \mathbb{R}^d$, $\theta \in (0,1]$ and $s \in [0,d]$, define

$$S^s_{\delta,\theta}(F) := \inf\Big\{ \sum_i (\operatorname{diam} U_i)^s : \{U_i\}_i \text{ is a cover of } F \text{ such that } \delta^{1/\theta} \leq \operatorname{diam} U_i \leq \delta \text{ for all } i \Big\}.$$

If there exists s such that $\frac{\log S^s_{\delta,\theta}(F)}{-\log \delta} \to 0$ as $\delta \to 0$, then we say that s is the *intermediate dimension* of F at θ, and write $s = \dim_\theta F$.

If the limit in the definition of box or intermediate dimension does not exist then we can use limsup and liminf to define upper and lower versions of the dimensions. The function $\theta \mapsto \dim_\theta F$ is always non-decreasing in θ, and is continuous for $\theta \in (0,1]$ but not generally at $\theta = 0$.

For many simple sets the intermediate dimensions have a simple form with at most finitely many points of non-differentiability, for example $\dim_\theta(\{1/n : n \geq 1\}) = \theta/(1+\theta)$ (see [3]). In contrast, the following characterisation shows that for general sets, the intermediate dimensions can have highly varied behaviour. Recall that the upper Dini derivative of a function $f: \mathbb{R} \to \mathbb{R}$ at x is given by

$$\mathrm{D}^+ f(x) := \limsup_{\varepsilon \to 0^+} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}.$$

Theorem 1. *(Banaji–Rutar [2]). Let $h: [0,1] \to [0,d]$ be any function. Then there exists $F \subset \mathbb{R}^d$ with $\dim_\theta F = h(\theta)$ if and only if h is non-decreasing, is continuous on $(0,1]$, and satisfies*

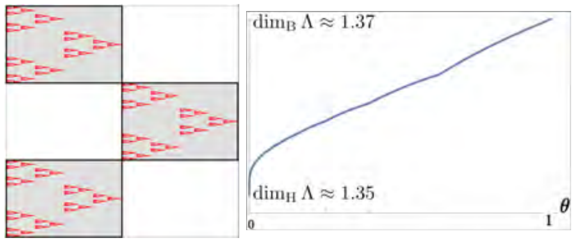
$$\mathrm{D}^+ h(\theta) \leq \frac{h(\theta)(d - h(\theta))}{d\theta} \qquad \text{for all } \theta \in (0,1).$$

It follows that if f is any non-decreasing Lipschitz function on $[0,1]$, there exist $F \subset \mathbb{R}$ and $a > 0$, $b \in \mathbb{R}$ such that $\dim_\theta F = af(\theta) + b$. In particular, the function can be non-differentiable at each point in a dense subset of $(0,1)$. A key step in the proof of Theorem 1 is the construction of a Cantor set with non-uniform subdivision ratios.

Bedford–McMullen carpets

An *iterated function system (IFS)* is a finite set of contraction maps $\{S_i: D \to D\}_{i=1}^N$, where $D \subset \mathbb{R}^d$ is closed. Given an IFS, there is a unique non-empty compact set Λ, called the *attractor*, satisfying $\Lambda = \bigcup_{i=1}^N S_i(\Lambda)$. Familiar fractals such as the middle-third Cantor set and the Sierpinski triangle are *self-similar sets*, which are the attractors of IFSs consisting of similarities (i.e. maps which contract distances by a constant ratio). The Hausdorff and box dimensions of self-similar sets always coincide, so we work with more general *self-affine sets*, where the contractions are affine. We work with the following particular class of sets, which have become a standard example in fractal geometry. Divide a square into an $m \times n$ grid, where $2 \leq m < n$, and choose a subset of the rectangles. Write $\gamma := \log n / \log m$. Consider the IFS of maps which send the square onto each of the chosen rectangles, preserving orientation. The attractor Λ is called a *Bedford–McMullen carpet*, after the authors who independently introduced these sets in 1984 and calculated their Hausdorff and box dimensions (which typically differ).

In [1], the author and Kolossváry calculate a precise formula (unfortunately too complicated to state here) for the intermediate dimensions of all Bedford–McMullen carpets. The proof uses tools from probability, dynamics and information theory, and involves explicitly constructing a cover using scales $\delta, \delta^\gamma, \delta^{\gamma^2}, \ldots, \delta^{\gamma^{L-1}}$ and $\delta^{1/\theta}, \delta^{1/(\gamma\theta)}, \ldots, \delta^{1/(\gamma^{L-1}\theta)}$ when $\gamma^{-L} < \theta < \gamma^{-(L-1)}$ (for $L \geq 1$). The intermediate dimensions have a more complicated form than previously observed for other classes of fractals.



A Bedford–McMullen carpet and the graph of its intermediate dimensions. Note that the graph has countably many phase transitions (see [1]) and is continuous on $[0,1]$ (see [3]).

Theorem 2. *(Banaji–Kolossváry [1]). Let Λ be a Bedford–McMullen carpet with $\dim_H \Lambda < \dim_B \Lambda$. Then the function $\theta \mapsto \dim_\theta \Lambda$ is strictly increasing. Moreover, for all integers $L \geq 1$, this function is real analytic and strictly concave on the interval $(\gamma^{-L}, \gamma^{-(L-1)})$, but is non-differentiable at $\theta = \gamma^{-L}$.*

Our formula also has useful applications. In particular, we use it to give a necessary condition for there to exist a Lipschitz bijection with a Lipschitz inverse between two Bedford–McMullen carpets.

FURTHER READING

[1] A. Banaji, I. Kolossváry, Intermediate dimensions of Bedford–McMullen carpets with applications to Lipschitz equivalence, Preprint 2021, arXiv:2111.05625.
[2] A. Banaji, A. Rutar, Attainable forms of intermediate dimensions, Ann. Fenn. Math. 47 (2022) 939–960.
[3] K. J. Falconer, J. M. Fraser, T. Kempton, Intermediate dimensions, Math. Z. 296 (2020) 813–830.

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