# Intermediate dimensions of Bedford–McMullen carpets

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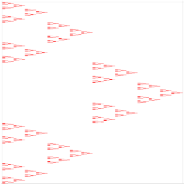


[1]Based on joint works with:
- Jonathan Fraser https://arxiv.org/abs/2104.15133, to appear in TAMS
- István Kolossváry https://arxiv.org/abs/2111.05625, Preprint.

# Hausdorff dimension

Alternative definition of Hausdorff dimension:

$$\dim_{\mathrm{H}} F = \inf\{\, s \geq 0 : \text{for all } \epsilon > 0 \text{ there exists a finite or countable cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } \sum_i |U_i|^s \leq \epsilon \,\}$$

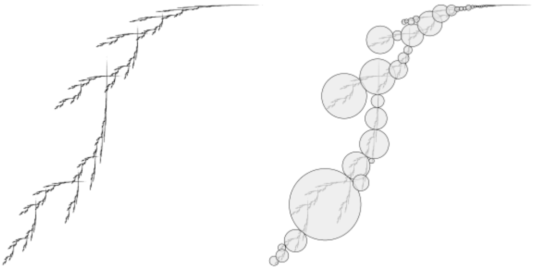


Figure: A cover using balls of different sizes. Picture by Jonathan Fraser.

# Box dimension

Alternative definition of box dimension:

$$\overline{\dim}_{\mathrm{B}} F = \inf\{\, s \geq 0 : \text{for all } \epsilon > 0 \text{ there exists } \delta_0 \in (0,1] \text{ such that for all } \delta \in (0,\delta_0) \text{ there exists a cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } |U_i| = \delta \text{ for all } i, \text{ and } \sum_i |U_i|^s \leq \epsilon \,\}.$$

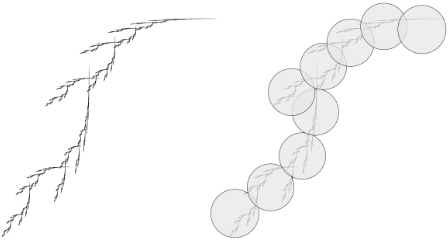


Figure: A cover using balls of the same size. Picture by Jonathan Fraser.

# Intermediate dimensions

Falconer, Fraser and Kempton ('20) defined the upper $\theta$-intermediate dimension for $\theta \in (0,1)$:

$$\overline{\dim}_\theta F = \inf\{\, s \geq 0 : \text{for all } \epsilon > 0 \text{ there exists } \delta_0 \in (0,1] \text{ such that for all } \delta \in (0,\delta_0) \text{ there exists a cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } \delta^{1/\theta} \leq |U_i| \leq \delta \text{ for all } i, \text{ and } \sum_i |U_i|^s \leq \epsilon \,\}.$$

- Always $\dim_{\mathrm{H}} F \leq \overline{\dim}_\theta F \leq \overline{\dim}_{\mathrm{B}} F$.
- The function $\theta \mapsto \overline{\dim}_\theta F$ is increasing, continuous for $\theta \in (0,1]$ but not necessarily at $\theta = 0$ (see $\Phi$-intermediate dimensions, B. '20).
- For more about the intermediate dimensions see Kenneth Falconer's talk.

# Iterated function systems (IFSs)

- Let $X \subset \mathbb{R}^n$ be compact and let $\{S_i \colon X \to X\}_{i \in I}$ be a finite IFS. There is a unique non-empty compact attractor $F$ satisfying $F = \bigcup_{i \in I} S_i(F)$.
- Throughout, we assume the open set condition (OSC): $\mathrm{Int}(X) \neq \varnothing$ and $\bigcup_{i \in I} S_i(\mathrm{Int}(X)) \subseteq \mathrm{Int}(X)$ with the union disjoint.
- If each $S_i$ is assumed to be a similarity map with contraction ratio $c_i$ then the Hausdorff and box dimensions of $F$ coincide with the unique $h \geq 0$ satisfying Hutchinson's formula

$$\sum_{i \in I} c_i^h = 1.$$

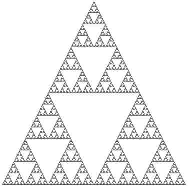


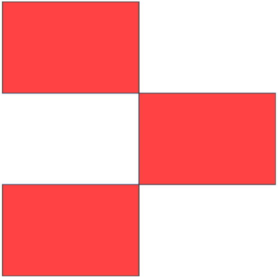
Figure: The Sierpinski gasket has Hausdorff dimension $\log 3 / \log 2$.

# Disparity of dimensions

Fractals with less homogeneity can have different Hausdorff and box dimensions. Two ways this can happen are if either:

1) There are infinitely many contractions in the IFS, or

2) The contractions are affine rather than similarities



# Infinite IFSs

- Consider an IFS consisting of a countably infinite number of similarity maps with contraction ratios $c_1, c_2, \ldots$ uniformly bounded below 1, satisfying the OSC.
- The limit set $F$ of a CIFS can be defined as the largest set (by inclusion) which satisfies
$$F = \bigcup_{i \in I} S_i(F).$$


First and second level cylinders for an infinitely generated self-similar set

- The Hausdorff dimension $h$ satisfies
$$\dim_{\mathrm{H}} F = \inf \left\{ t \geq 0 : \sum_{i \in I} c_i^t \leq 1 \right\}.$$

# Box and intermediate dimensions

Assuming the compact set $X$ satisfies the cone condition, and $P$ is the set of fixed points of the contractions,

- $\overline{\dim}_{\mathrm{B}} F = \max\{\dim_{\mathrm{H}} F, \overline{\dim}_{\mathrm{B}} P\}$ (Mauldin – Urbański, '99, TAMS)
- $\overline{\dim}_\theta F = \max\{\dim_{\mathrm{H}} F, \overline{\dim}_\theta P\}$ (B. – Fraser, to appear in TAMS)

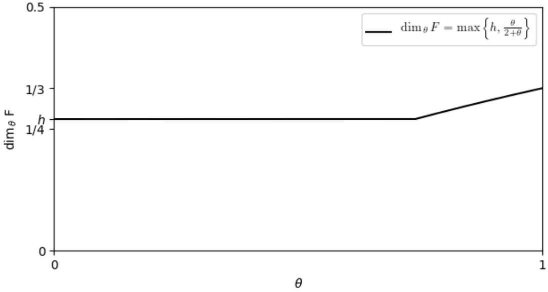
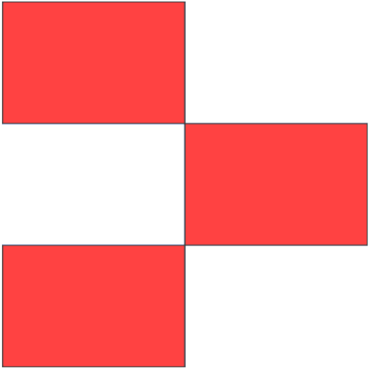



Figure: Intermediate dimensions when $P = \{\, k^{-2} : k \in \mathbb{N} \,\}$

- The formulae in blue hold even in when the contractions are conformal, which has applications to parabolic systems and continued fraction sets.

# Bedford–McMullen carpets

Divide a square into an $m \times n$ grid, $2 \leq m < n$.



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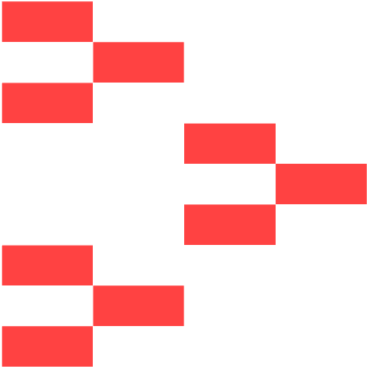

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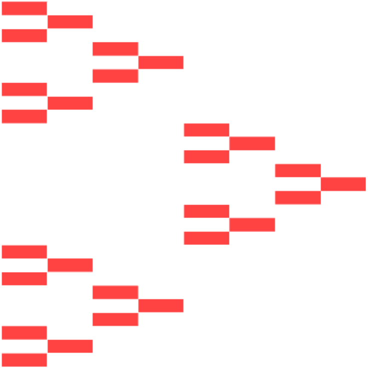

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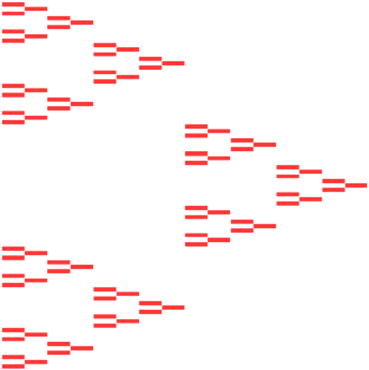

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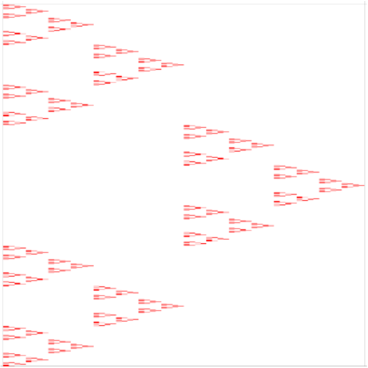


# Hausdorff and box dimensions

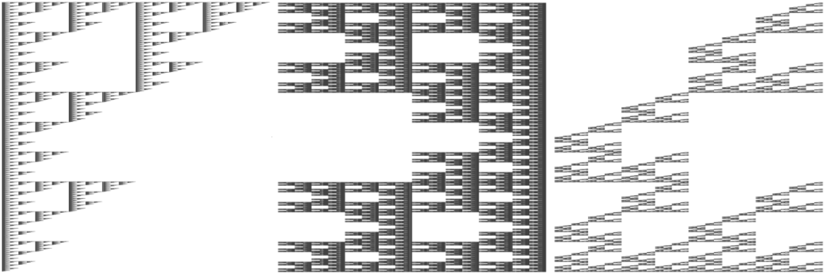
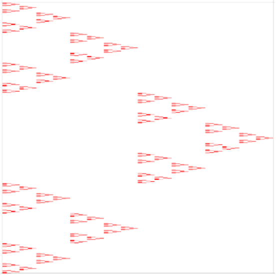


Figure: Three different Bedford–McMullen carpets. Picture by Jonathan Fraser

Bedford ('84) and McMullen ('84) independently calculated their Hausdorff and box dimensions.
Throughout, we assume that $\Lambda$ has non-uniform vertical fibres, or equivalently that $\dim_{\mathrm{H}} \Lambda < \dim_{\mathrm{B}} \Lambda$.

# Graph of the intermediate dimensions



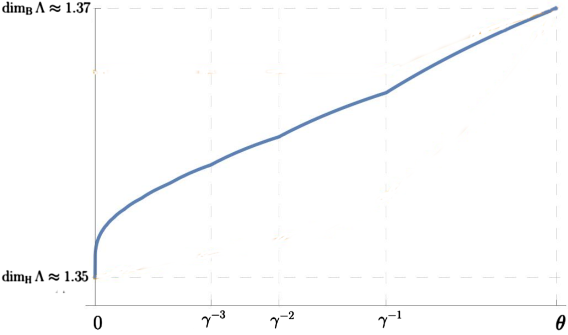



Figure: Here, $\gamma := \log n / \log m$

# Formula for the intermediate dimensions (B. – Kolossváry, '21+)

- Let $M := \#$ non-empty columns, $N_i := \#$ maps in $i$th non-empty column, $N := N_1 + \cdots + N_M$. Define the Legendre transform

$$I(t) := \sup_{\lambda \in \mathbb{R}} \left\{ \lambda t - \log \left( \frac{1}{M} \sum_{\hat{\jmath}=1}^{M} N_{\hat{\jmath}}^{\lambda} \right) \right\}.$$

- For $s \in \mathbb{R}$, define the function $T_s \colon \mathbb{R} \to \mathbb{R}$ by

$$T_s(t) := \left( s - \frac{\log M}{\log m} \right) \log n + \gamma I(t).$$

- For $\ell \in \mathbb{N}$, write $T_s^\ell := \underbrace{T_s \circ \cdots \circ T_s}_{\ell \text{ times}}$, and $T_s^0$ is the identity. Define

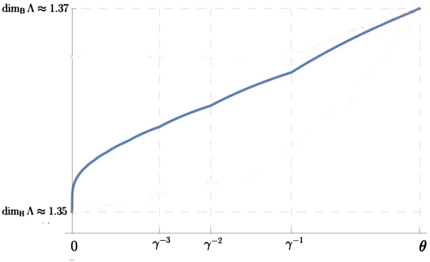
$$t_\ell(s) := T_s^{\ell-1}\left( \left( s - \frac{\log M}{\log m} \right) \log n \right).$$

- For fixed $\theta \in (0,1)$ let $L = L(\theta) \in \mathbb{N}$ be such that $\gamma^{-L} < \theta \leq \gamma^{-(L-1)}$. Then $\dim_\theta \Lambda$ is the unique solution $s = s(\theta) \in (\dim_{\mathrm{H}} \Lambda, \dim_{\mathrm{B}} \Lambda)$ to the equation

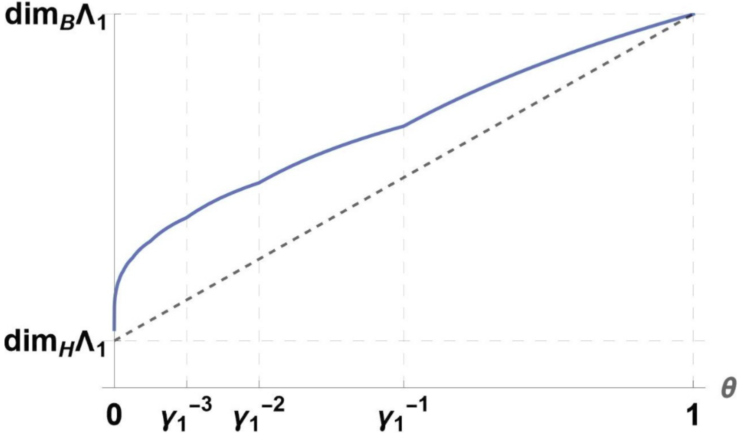
$$\gamma^L \theta \log N - (\gamma^L \theta - 1) t_L(s) + \gamma(1 - \gamma^{L-1}\theta)(\log M - I(t_L(s))) - s \log n = 0.$$

# Intermediate dimensions of Bedford–McMullen carpets

- Phase transitions at negative integer powers of $\log n / \log m$.
- Real analytic and strictly concave between phase transitions
- Strictly increasing
- Right derivative tends to $\infty$ as $\theta \to 0$
- Continuous for $\theta \in [0, 1]$ (proved by Falconer – Fraser – Kempton ('20), used by Burrell – Falconer – Fraser ('21) to prove results on the box dimension of orthogonal projections of carpets)



# Different possible shapes of the graph



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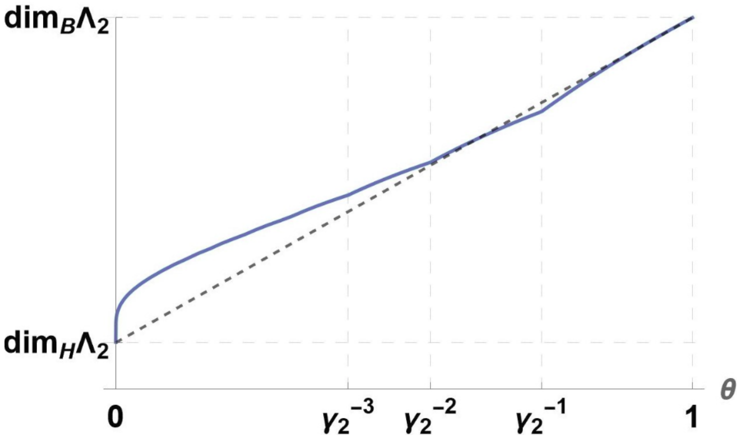

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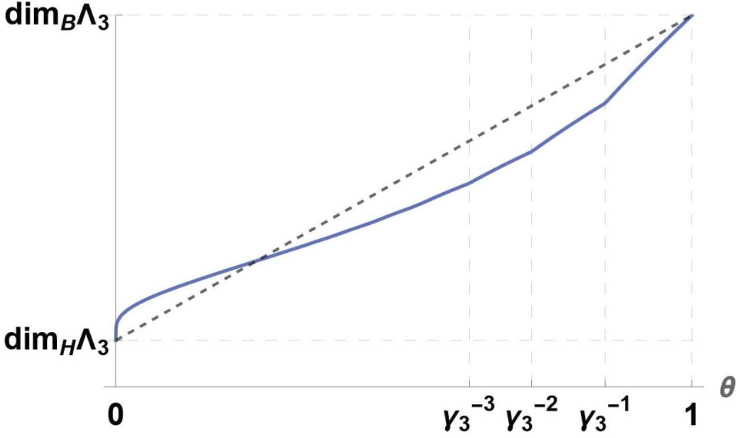


# Multifractal analysis

- Let $\nu$ be the uniform Bernoulli measure supported on a Bedford–McMullen carpet, satisfying

$$\nu(A) = \sum_{i=1}^{N} \frac{1}{N} \nu(S_i^{-1}A) \text{ for all Borel sets } A \subset \mathbb{R}^2.$$

  where $N$ is the total number of contractions.
- Jordan and Rams ('11) computed the multifractal spectrum of $\nu$,

$$f_\nu(\alpha) \coloneqq \dim_{\mathrm{H}}\{x \in \operatorname{supp} \nu : \dim_{\mathrm{loc}}(\nu, x) = \alpha\},$$

  building on work of King ('95).

**Theorem (B. – Kolossváry, '21+)**

If $\Lambda$, $\Lambda'$ are Bedford–McMullen carpets with non-uniform vertical fibres, then the intermediate dimensions are equal for all $\theta$ if and only if the corresponding uniform Bernoulli measures have the same multifractal spectra.

# Bi-Lipschitz equivalence

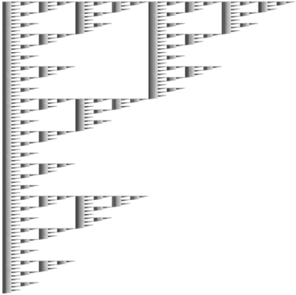
If $f\colon \Lambda \to \Lambda'$ is bi-Lipschitz then $\dim_\theta \Lambda = \dim_\theta \Lambda'$ for all $\theta$.

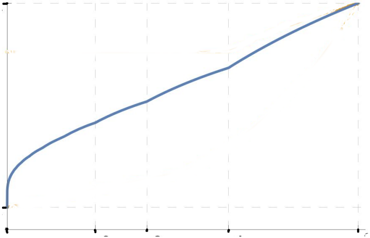
## Corollary

If carpets $\Lambda$ and $\Lambda'$ with non-uniform vertical fibres are bi-Lipschitz equivalent then their uniform Bernoulli measures have the same multifractal spectra.

This improves a result of Rao, Yang and Zhang ('21+).

# Thank you for listening!





Questions welcome