(1) * **A double integral**

Let $f\colon [0,2]\times[0,1]\to\mathbb{R}$, $f(x,y)=xe^y$. Note that $([0,2]\times[0,1],\mathcal{B}([0,2]\times[0,1]),\lambda_2)$ is a finite, hence σ-finite, measure space. Note that f is continuous, hence $(\mathcal{B}([0,2]\times[0,1]),\mathcal{B}(\mathbb{R}))$-measurable. Since $f(x,y)\geq 0$ for all $(x,y)\in[0,2]\times[0,1]$, we can apply Tonelli's theorem. (Alternative: since $|f(x,y)|\leq 2e$ for all $(x,y)\in[0,2]\times[0,1]$, it holds that $\int_{[0,2]\times[0,1]}|f(x,y)|d\lambda_2(x,y)\leq\int_{[0,2]\times[0,1]}2ed\lambda_2(x,y)=4e<\infty$, so we can apply Fubini's theorem.) Therefore

$$\int_{[0,2]\times[0,1]}f(x,y)d\lambda_2(x,y)=\int_{[0,2]}\int_{[0,1]}f(x,y)d\lambda(y)d\lambda(x)=\int_0^2 xd\lambda(x)\int_0^1 e^y d\lambda(y)=\Big[\frac{x^2}{2}\Big]_0^2[e^y]_0^1=2(e-1).$$

We were allowed to replace the Lebesgue integrals by Riemann integrals because $x\mapsto x$ and $y\mapsto e^y$ are continuous.

(2) **Assumptions of Tonelli / Fubini**

Let $f\colon(0,1)\times(0,1)\to\mathbb{R}$, $f(x,y)=\frac{x^2-y^2}{(x^2+y^2)^2}$. **Note: a previous version of the question said $f\colon(0,1)\to(0,1)$, this was clearly a typo, apologies.**

(a) For integers $n\geq 1$, define the disjoint sets $R_n\coloneqq[2^{-n},2^{-n}+2^{-(n+1)}]\times[0,2^{-(n+1)}]$. Note that $x\geq y$ for all $(x,y)\in R_n$, and $f\geq 0$ on R_n. Moreover, when $x\geq y$, when we keep x fixed and increase y, $f(x,y)$ clearly decreases. Furthermore, by the quotient rule, on R_n it holds that

$$\frac{\partial}{\partial x}f(x,y)=\frac{2x(x^2+y^2)-(x^2-y^2)\cdot 2(x^2+y^2)\cdot(2x)}{(x^2+y^2)^4}=\frac{2x(y^2-x^2)}{(x^2+y^2)^3}<0.$$

Therefore $f(x,y)\geq f(2^{-n},2^{-(n+1)})=3\cdot(2^{2n})/5$ for all $(x,y)\in R_n$. It follows that

$$\int_{(0,1)^2}|f(x,y)|d\lambda_2(x,y)\geq\sum_{n=1}^{\infty}\int_{R_n}f(x,y)d\lambda_2(x,y)\geq\sum_{n=1}^{\infty}2^{-2(n+1)}\cdot\frac{3}{5}2^{2n}=\sum_{n=1}^{\infty}\frac{3}{20}=\infty.$$

(b) We can't apply Tonelli's theorem because f changes sign on $(0,1)\to(0,1)$, and we can't apply Fubini's theorem either because of part (a).

(c) Using the hints,

$$\int_{[0,1]}\int_{[0,1]}f(x,y)d\lambda(x)d\lambda(y)=\int_{[0,1]}\Big[\frac{-x}{x^2+y^2}\Big]_0^1 d\lambda(y)=\int_{[0,1]}\Big(-\frac{1}{1+y^2}\Big)d\lambda(y)=[-\arctan(y)]_0^1=-\frac{\pi}{4}.$$

(d) A similar calculation (or using (c)) together with the fact that $f(x,y)=-f(y,x))$ gives that

$$\int_{[0,1]}\int_{[0,1]}f(x,y)d\lambda(y)d\lambda(x)=\frac{\pi}{4}.$$

(3) **Riemann vs Lebesgue integrability**

(a) For all $x\in[0,1]$ and $\delta>0$, let n be large enough that $2^{-n}<\delta$, so we can find a dyadic rational $y\in(0,1)$ of the form $m/2^n$ such that $|x-y|<\delta$. We can also find an irrational number $z\in(0,1)$ such that $|x-y|<\delta$. Then $f(y)=1$ while $f(z)=0$, so regardless of whether or not x is a dyadic rational, $f(y)$ and $f(z)$ cannot both be within distance $1/3$ of $f(x)$. Therefore for all $x\in[0,1]$, f is *not* continuous at x.

(b) Case 1): suppose $x\in(0,1]$ is rational. Then we can find a sequence of irrational numbers (x_n) in $(0,1)$ converging to x. But $g(x_n)=0\to 0<g(x)$, so g is not continuous at x.
Case 2): Suppose $x=0$ or $x\in(0,1)\setminus\mathbb{Q}$, then $g(x)=0$, and for all $\varepsilon>0$ we can find $N\in\mathbb{N}$ large enough that $1/N<\varepsilon$. Then since there are only finitely many rationals that can be written with denominator at most N, we can find $\delta>0$ such that $(x-\delta,x+\delta)$ does not contain any of these points, and so $0\leq g(y)<1/N<\varepsilon$ for all $y\in(x-\delta,x+\delta)\cap[0,1]$.

(c) The Lebesgue measure of the set of $x \in [0,1]$ at which x is discontinuous is $\lambda([0,1]) = 1 > 0$, so by Lebesgue's Criterion for Riemann Integrability, f is not Riemann integrable.

(d) Since g is continuous at all but a countable (hence zero-Lebesgue-measure) set, g is Riemann integrable by Lebesgue's Criterion for Riemann Integrability. (In fact, we can see that by the density of irrational numbers in $(0,1)$, for any partition the lower Darboux sum of g is 0, so $\int_0^1 g(x)dx = 0$.)

(e) f is the indicator function of a countable (hence zero-Lebesgue-measure) set, so f is a simple function, hence Borel measurable and Lebesgue integrable, with $\int_{[0,1]} f d\lambda = 0$.

(f) For $a < 0$, $g^{-1}((a,\infty)) = [0,1] \in \mathcal{B}([0,1])$, while if $a \geq 0$ then $g^{-1}((a,\infty))$ is countable so is again a Borel subset of $[0,1]$. Therefore g is $(\mathcal{B}([0,1]), \mathcal{B}(\mathbb{R}))$-measurable. But g is also a bounded, Riemann-integrable function on $[0,1]$, so g must be Lebesgue integrable, with $\int_{[0,1]} g d\lambda = \int_0^1 g(x)dx = 0$.

(4) **independence and expectation**

(a) Take $([0,1], \mathcal{B}([0,1]), \lambda)$, and

$$X(\omega) = \sqrt{\omega}\mathbf{1}_{(0,3/4]}(\omega), \quad Y(\omega) = \frac{1}{\sqrt{\omega}}\mathbf{1}_{(0,3/4]}(\omega).$$

Then

$$\lambda(X \in [0, 1/\sqrt{2}], Y \in (0,\infty)) = \lambda((0,1/2] \cap (0,3/4]) = 1/2 \neq 3/8 = \lambda((0,1/2])\lambda((0,3/4]),$$

so X, Y are not independent. But

$$\int_0^1 XY d\lambda = \int_0^{3/4} 1 d\lambda = 3/4,$$
$$\int_0^1 X d\lambda = \int_0^{3/4} \sqrt{x}dx = [2x^{3/2}/3]_0^{3/4} = 2(3/4)^{3/2}/3,$$
$$\int_0^1 Y d\lambda = \int_0^{3/4} 1/\sqrt{x}dx = [2\sqrt{x}]_0^{3/4} = 2(3/4)^{1/2}.$$

Thus $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y) = 3/4$.

(b) Choose

$$f(x) = \mathbf{1}_A(x),\ A \in \mathcal{B}(\mathbb{R}), \qquad g(x) = \mathbf{1}_B(x),\ B \in \mathcal{B}(\mathbb{R}).$$

Then

$$\mathbb{P}(X \in A,\ Y \in B) = \mathbb{E}[f(X)g(Y)] = \mathbb{E}[f(X)]\,\mathbb{E}[g(Y)] = \mathbb{P}(X \in A)\,\mathbb{P}(Y \in B),$$

so we have independence.

(5) **an integrability criteria for non-negative functions** When $n \leq t \leq n+1$ we clearly have

$$\mathbb{P}(f > n+1) \leq \mathbb{P}(f > t) \leq \mathbb{P}(f > n).$$

Integrating gives

$$\mathbb{P}(f > n+1) \leq \int_n^{n+1} \mathbb{P}(f > t)dt \leq \mathbb{P}(f > n).$$

Summing,

$$\sum_{n=0}^{\infty} \mathbb{P}(f > n+1) \leq \int_0^{\infty} \mathbb{P}(f \geq t)dt \leq \sum_{n=0}^{\infty} \mathbb{P}(f > n).$$

Thus $\int_0^\infty \mathbb{P}(f \geq t)dt < \infty$ if and only if $\sum_{n=1}^\infty \mathbb{P}(f > n) < \infty$. But the Layer Cake Formula says that

$$\mathbb{E}(f) = \int_0^\infty \mathbb{P}(f \geq t)dt,$$

and the result is proved.

(6) **the n-th moment of a non-negative random variable**
The Layer Cake Formula applied to f^p gives that

$$\mathbb{E}(f^p) = \int_{[0,\infty)} \mathbb{P}(f^p > t) d\lambda(t).$$

By the Monotone Convergence Theorem it follows that

$$\mathbb{E}(f^p) = \lim_{N\to\infty} \int_{[0,N]} \mathbb{P}(f^p > t) d\lambda(t).$$

But the function $t \mapsto \mathbb{P}(f^p > t)$ is decreasing, so each discontinuity point gives a distinct open interval (between the left and right endpoints) which contains a rational, so since the rationals are countable this function must have at most countably many discontinuity points. Therefore by Lebesgue's Criterion for Riemann Integrability we can move between the Lebesgue and Riemann integrals, and then using the substitution $t = u^p$ gives

$$\mathbb{E}(f^p) = \lim_{N\to\infty} \int_0^N \mathbb{P}(f^p > t) dt = \lim_{N\to\infty} \int_0^{N^{1/p}} \mathbb{P}(f > u) p u^{p-1} du.$$

Thus

$$\mathbb{E}(f^p) = \int_0^\infty \mathbb{P}(f > u) p u^{p-1} d\lambda(u).$$

(7) **Concave Jensen**
Applying Jensen's inequality to the convex function $-g$ gives $(-g)(\mathbb{E}f) \leq \mathbb{E}(-g)(f) = -\mathbb{E}g(f)$, hence $g(\mathbb{E}f) \geq \mathbb{E}(g(f))$.