Please hand in your answer to Question (1)* by email before the start of the exercise session, or on paper at the start of the exercise session.
You do not need to hand in the rest of your solutions (unless you are unable to attend the exercise session), but be prepared to present one of the ones you have completed (I will choose which one) during the exercise session. If you are unable to attend the exercise session, please email me all the questions you have attempted by 10am on Monday.

(1) * **A double integral**
Calculate
$$\int_{[0,2]\times[0,1]} xe^y d\lambda_2(x,y).$$
If you use a result from the course, justify carefully why the assumptions of that result are satisfied.

(2) **Assumptions of Tonelli / Fubini**
Let $f\colon (0,1) \to (0,1)$ be defined by
$$f(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2}.$$
(a) Prove that $\int_{(0,1)^2} |f(x,y)| d\lambda_2(x,y) = \infty$.
(b) Can we use either Tonelli's theorem of Fubini's theorem to calculate the iterated integrals? Why / why not?
(c) Calculate
$$\int_0^1 \left(\int_0^1 f(x,y) dx \right) dy.$$
Hint: for any constant c it holds that $\frac{d}{dx}\left(\frac{-x}{x^2+c^2}\right) = f(x,c)$, and $\frac{d}{dy}(\arctan(y)) = \frac{1}{1+y^2}$.
(d) Calculate
$$\int_0^1 \left(\int_0^1 f(x,y) dy \right) dx.$$

(3) **Riemann vs Lebesgue integrability**
Recall that a real number x is called a *dyadic rational* is there are integers n, m such that $x = \frac{m}{2^n}$. Let $f, g\colon [0,1] \to \mathbb{R}$ be defined by
$$f(x) := \begin{cases} 1 & \text{if } x \text{ is a dyadic rational,} \\ 0 & \text{otherwise,} \end{cases} \qquad g(x) := \begin{cases} \frac{1}{q} & \text{if } x = p/q > 0 \text{ where } p,q \in \mathbb{N} \text{ cannot be reduced,} \\ 0 & \text{if } x \notin \mathbb{Q} \text{ or } x = 0, \end{cases}$$
(note that g is called the *popcorn function* or *Thomae's function*).
(a) Find precisely the set of $x \in [0,1]$ at which f is continuous.
(b) Find precisely the set of $x \in [0,1]$ at which g is continuous.
(c) Is f Riemann integrable?
(d) Is g Riemann integrable?
(e) Is f Lebesgue integrable?
(f) Is g Lebesgue integrable?
Justify your answers, using results from Section 8.6.2 of the book where appropriate.

(4) **independence and expectation**
From the course Probability Theory 1 we know that the independence of two integrable random variables $X, Y\colon \Omega \to \mathbb{R}$ implies that their product is integrable, and it holds that $\mathbb{E}(XY) = (\mathbb{E}X)\cdot(\mathbb{E}Y)$.
(a) Find random variables $X, Y\colon \Omega \to \mathbb{R}$ on some probability space that are not independent, but $\mathbb{E}(XY) = (\mathbb{E}X)\cdot(\mathbb{E}Y)$.

(b) Assume that $\mathbb{E}f(X)g(Y) = \mathbb{E}f(X)\mathbb{E}g(Y)$ holds for all bounded, $(\mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}))$-measurable functions $f, g\colon \mathbb{R} \to \mathbb{R}$. Prove that X and Y are independent.

(5) **an integrability criteria for non-negative functions**
Show that for any non-negative random variable $f : \Omega \to [0, \infty)$ it holds that

$$\mathbb{E}f < \infty \Longleftrightarrow \sum_{n=1}^{\infty} \mathbb{P}(f > n) < \infty.$$

(6) **the n-th moment of a non-negative random variable**
Let $f\colon \Omega \to [0, \infty)$ be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. Show that for every $p \geq 1$ it holds that

$$\mathbb{E}f^p = \int_0^\infty pt^{p-1}\mathbb{P}(f > t)dt.$$

(7) **Concave Jensen**
Assume that $g\colon \mathbb{R} \to \mathbb{R}$ is concave, and f is an integrable random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. Show that it holds that $g(\mathbb{E}f) \geq \mathbb{E}g(f)$.
Hint: This is very easy!