

(1) **Hölder or Jensen**

(a) Hölder: $\mathbb{E}|fg| \leq (\mathbb{E}|f|^p)^{1/p}(\mathbb{E}|g|^q)^{1/q}$ and take $g \equiv 1$.

Alternative (Jensen): For $N \in \mathbb{N}$ let $f_N(\omega) := \min\{|f(\omega)|, N\}$, so $\mathbb{E}(f_N) \leq N < \infty$. Then by Jensen applied to the convex function $x \mapsto |x|^p$ it holds that $(\mathbb{E}(f_N))^p \leq \mathbb{E}(f_N^p)$. By the monotone convergence theorem it follows that $(\mathbb{E}|f|)^p \leq \mathbb{E}(|f|^p)$, and the result follows.

(b) Apply (1a): $\mathbb{E}|g| \leq (\mathbb{E}|g|^r)^{1/r}$ with $g = |f|^p$ and $r = q/p$. Then

$$(\mathbb{E}(|f|^p))^{1/p} \leq (\mathbb{E}|f|^{p \cdot \frac{q}{p}})^{\frac{p}{q} \cdot \frac{1}{p}} = (\mathbb{E}|f|^q)^{1/q}.$$

(2) **Countable measure spaces**

(a) Clearly $\mu(\emptyset) = \sum_{n \in \emptyset} p_n = 0$. For all $A \subseteq \mathbb{N}$ we have $\mu(A) = \sum_{n \in A} p_n \geq 0$ since each $p_n \geq 0$. To show countable additivity, let A_1, A_2, \dots be pairwise-disjoint. Then

$$\mu\left(\bigcup_{k \geq 1} A_k\right) = \sum_{n \in \bigcup_{k \geq 1} A_k} p_n = \sum_{k \geq 1} \left(\sum_{n \in A_k} p_n\right) = \sum_{k \geq 1} \mu(A_k),$$

where we used pairwise-disjointness for the middle equality.

(b) For each n let $q_n := \nu(\{n\})$. Then for $A \subseteq \mathbb{N}$, we have $\nu(A) = \sum_{n \in A} q_n$ by countable additivity of ν .

(c) For $N \in \mathbb{N}$ define $f_N: \mathbb{N} \rightarrow [0, \infty)$ by

$$f_N(n) = \begin{cases} f(n), & \text{if } n \leq N, \\ 0, & \text{if } n > N. \end{cases}$$

Using the definition of the Lebesgue integral of a simple function and then monotone convergence,

$$\sum_{n \leq N} p_n f(n) = \int_{\mathbb{N}} f_N d\mu \xrightarrow{N \rightarrow \infty} \int_{\mathbb{N}} f d\mu.$$

But $\sum_{n \leq N} p_n f(n) \rightarrow \sum_{n \in \mathbb{N}} p_n f(n)$ as $N \rightarrow \infty$, so the result follows.

(3) **Useful inequalities**

(a) We use the measure space $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ from problem (2a) with weights $p_n = 1$ for all $n \in \mathbb{N}$. Let $f(n) = a_n$ and $g(n) = b_n$ for all $n \in \mathbb{N}$. Then Hölder's inequality with the definition of the integral from (2c) becomes precisely the inequality we want to prove.

(b) We apply (3a) with sequences $a_1, \dots, a_n, 0, 0, \dots$ and $\underbrace{1, \dots, 1}_{n \text{ times}}, 0, 0, \dots$ to get

$$\sum_{k=1}^n a_k \leq \left(\sum_{k=1}^n a_k^p\right)^{1/p} \cdot n^{1/q}.$$

Since $x \mapsto x^p$ (for $x \geq 0$) is increasing, raising both sides of the inequality to the power p gives the result.

(c)

$$\left(\sum_{k=1}^n a_k\right)^p = \left(a_1 + \sum_{k=2}^n a_k\right)^p \leq a_1^p + \left(\sum_{k=2}^n a_k\right)^p \leq a_1^p + a_2^p + \left(\sum_{k=3}^n a_k\right)^p \leq \dots \leq \sum_{k=1}^n a_k^p.$$

(4) **convex functions**

For $n \in \mathbb{N}$ let $a_n := n(g(x + \frac{1}{n}) - g(x))$. Then

$$\begin{aligned} a_n - a_{n+1} &= ng\left(x + \frac{1}{n}\right) - ng(x) - (n+1)g\left(x + \frac{1}{n+1}\right) + (n+1)g(x) \\ &= (n+1)\left[\frac{n}{n+1}g\left(x + \frac{1}{n}\right) + \frac{1}{g(x)} - g\left(\frac{1}{n+1}x + \left(1 - \frac{1}{n+1}\right)\left(x + \frac{1}{n}\right)\right)\right] \geq 0, \end{aligned}$$

where the final inequality was by the convexity inequality $g(\theta x + (1 - \theta)y) \leq \theta g(x) + (1 - \theta)g(y)$ with $y = x + \frac{1}{n}$, $\theta = \frac{1}{n+1}$, $1 - \theta = \frac{n}{n+1}$.

(5) **Sharpness of Minkowski's inequality**

Take $([0, 1], \mathcal{B}([0, 1]), \lambda)$, $f = \mathbf{1}_{[0, 1/2]}$, $g = \mathbf{1}_{[1/2, 1]}$. Then for $0 < p < 1$,

$$\left(\int_{[0, 1]} |f + g|^p d\lambda\right)^{1/p} = 1 = 2^{\frac{1}{p}-1} \left(\left(\frac{1}{2}\right)^{1/p} + \left(\frac{1}{2}\right)^{1/p}\right) = 2^{\frac{1}{p}-1} \left(\left(\int_{[0, 1]} |f|^p d\lambda\right)^{1/p} + \left(\int_{[0, 1]} |g|^p d\lambda\right)^{1/p}\right).$$

(6) **True or false**

(a) False. For a counterexample let $p = 1 < 2 = q$ and $f = \frac{1}{x} \mathbf{1}_{[1, \infty)}(x)$. Then

$$\left(\int_{\mathbb{R}} |f|^q d\lambda\right)^{\frac{1}{q}} = \left(\int_1^{\infty} \frac{1}{x^2} dx\right)^{1/2} = \sqrt{\left[-\frac{1}{x}\right]_1^{\infty}} = 1 < \infty = \int_1^{\infty} \frac{1}{x} dx = \left(\int_{\mathbb{R}} |f|^p d\lambda\right)^{\frac{1}{p}}.$$

Note that (1b) does not apply because $\lambda(\mathbb{R}) = \infty$.

(b) True by Markov's inequality:

$$\mathbb{P}(|g| \geq \lambda) = \mathbb{P}(e^{|g|} \geq e^\lambda) \leq e^{-\lambda} \mathbb{E}(e^{|g|}).$$

(c) False. For a counterexample, take $((0, 1), \mathcal{B}(0, 1), \lambda)$ and $f(x) = x^{-3/4}$, noting that f is continuous hence Borel measurable. Then

$$\mathbb{E}|f| = \mathbb{E}f = \int_0^1 x^{-3/4} dx = [4x^{1/4}]_0^1 = 4,$$

so f is integrable. But

$$\text{Var}(f) \geq \int_0^{4^{-4/3}} (x^{-3/4} - 4)^2 dx = \int_0^{4^{-4/3}} (x^{-3/2} + 16 - 8x^{-3/4}) dx = [-2x^{-1/2} + 16x - 32x^{1/4}]_0^{4^{-4/3}} = \infty.$$

(7) **convergence in probability**

(a) Fix $\varepsilon > 0$. Since $f_n \xrightarrow{\mathbb{P}} f$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ it holds that $\mathbb{P}(|f_n - f| > \varepsilon/2) \leq \varepsilon$. Then for all $n, m \geq N$,

$$\mathbb{P}(|f_n - f_m| > \varepsilon) \leq \mathbb{P}(|f_n - f| > \varepsilon/2) + \mathbb{P}(|f_n - f| > \varepsilon/2) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

So (f_n) is Cauchy in probability.

(b) As in (7a), for $N, n \in \mathbb{N}$,

$$\mathbb{P}(|f - g| > 2/N) \leq \mathbb{P}(|f - f_n| > 1/N) + \mathbb{P}(|f_n - g| > 1/N) \xrightarrow{n \rightarrow \infty} 0,$$

so $\mathbb{P}(|f - g| > 2/N) = 0$ for all N . Thus

$$1 \geq \mathbb{P}(f = g) = 1 - \mathbb{P}(|f - g| > 0) = 1 - \bigcup_{N=1}^{\infty} \mathbb{P}(|f - g| > 2/N) \geq 1 - \sum_{N=1}^{\infty} 0 = 1$$

so there is equality throughout.

(c) Let $1 \leq n_1 < n_2 < \dots$ be an arbitrary subsequence. Since $f_n \xrightarrow{\mathbb{P}} f$, by Theorem 9.2.4 (4) we can find a subsequence $n_{k_1} < n_{k_2} < \dots$ such that $f_{n_k} \xrightarrow[k \rightarrow \infty]{a.s.} f$. Since $g_n \xrightarrow{\mathbb{P}} g$, we can find a further subsequence $n_{k_{j_1}} < n_{k_{j_2}} < \dots$ such that $g_{n_{k_j}} \xrightarrow[j \rightarrow \infty]{a.s.} g$, and note that also $f_{n_{k_j}} \xrightarrow[j \rightarrow \infty]{a.s.} f$.

Now $\lambda f_n + \mu g_n \xrightarrow{\mathbb{P}} \lambda f + \mu g$ follows by the backward implication of Theorem 9.2.4 (4), since

$$\lambda f_{n_{k_j}} + \mu g_{n_{k_j}} \xrightarrow[j \rightarrow \infty]{a.s.} \lambda f + \mu g.$$

Similarly, $f_n g_n \xrightarrow{\mathbb{P}} fg$ holds since

$$f_{n_{k_j}} g_{n_{k_j}} \xrightarrow[j \rightarrow \infty]{a.s.} fg.$$