Please hand in your answer to Question (1)* by email before the start of the exercise session, or on paper at the start of the exercise session.
You do not need to hand in the rest of your solutions (unless you are unable to attend the exercise session), but be prepared to present one of the ones you have completed (I will choose which one) during the exercise session. If you are unable to attend the exercise session, please email me all the questions you have attempted by 10am on the day of the exercise session.
Note: this sheet is a little longer than most sheets will be because you have two weeks to complete this sheet.

(1) * **Hölder or Jensen**
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $f\colon \Omega \to \mathbb{R}$ a random variable. Use Hölder's or Jensen's inequality to show that
 (a) for $1 < p < \infty$ it holds that $\mathbb{E}|f| \leq (\mathbb{E}|f|^p)^{\frac{1}{p}}$,
 (b) for $0 < p < q < \infty$ one has that $(\mathbb{E}|f|^p)^{\frac{1}{p}} \leq (\mathbb{E}|f|^q)^{\frac{1}{q}}$.

(2) **Countable measure spaces**
Consider the measurable space $(\mathbb{N}, 2^{\mathbb{N}})$.
 (a) Suppose for each $n \in \mathbb{N}$ we associate a number $p_n \in [0, \infty]$. Given $A \subseteq \mathbb{N}$ let
$$\mu(A) \coloneqq \sum_{n \in A} p_n \quad \text{with} \sum_{n \in \varnothing} p_n \coloneqq 0.$$
 Show that μ is a measure.
 (b) Given any measure ν on $(\mathbb{N}, 2^{\mathbb{N}})$, show that for each $n \in \mathbb{N}$ there is some $q_n \in [0, \infty]$ such that for every $A \subseteq \mathbb{N}$ it holds that
$$\nu(A) = \sum_{n \in A} q_n \quad (\text{where} \sum_{n \in \varnothing} q_n \coloneqq 0).$$
 (c) Letting μ be the measure from (2a), and given $f\colon \mathbb{N} \to [0, \infty)$, show that
$$\int_{\mathbb{N}} f d\mu = \sum_{n \in \mathbb{N}} p_n f(n).$$

(3) **Useful inequalities**

 (a) Use Hölder's inequality on an appropriate measure space to show that for all real sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, and $1 < p, q < \infty$ with $1/p + 1/q = 1$, it holds that
$$\sum_{n\in\mathbb{N}} |a_n b_n| \leq \left(\sum_{n\in\mathbb{N}} |a_n|^p\right)^{1/p} \left(\sum_{n\in\mathbb{N}} |b_n|^q\right)^{1/q}.$$
 (b) Use (3a) to show that for each $n \in \mathbb{N}$ and $(a_k)_{k=1}^n \subseteq [0, \infty)$ and $p > 1$ it holds that
$$\left(\sum_{k=1}^n a_k\right)^p \leq n^{p-1} \left(\sum_{k=1}^n a_k^p\right).$$
 (**Hint:** $b_1 = \cdots = b_n = 1$.)
 (c) In the proof of Minkowski's inequality we have shown that for $p \in (0, 1)$ and $a, b \geq 0$ it holds that
$$(a + b)^p \leq a^p + b^p.$$
 Show by induction that for $a_1, \ldots, a_n \in [0, \infty)$ and $p \in (0, 1)$ it holds that
$$\left(\sum_{k=1}^n a_k\right)^p \leq \left(\sum_{k=1}^n a_k^p\right).$$

(4) **convex functions**
For a convex function $g\colon \mathbb{R} \to \mathbb{R}$ and fixed $x \in \mathbb{R}$, show that the sequence

$$(n(g(x+\tfrac{1}{n}) - g(x)))_{n=1}^{\infty}$$

is decreasing.

(5) **Sharpness of Minkowski's inequality**
Find an example that shows that the constant $c_p = 2^{\frac{1}{p}-1}$ in Minkowski's inequality is sharp for $0 < p < 1$.

(6) **True or false**
Determine which of the following assertions are true (or false) and provide a proof (or a counterexample):

(a) For every random variable f on $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, and every $0 < p < q < \infty$, one has that

$$\left(\int_{\mathbb{R}} |f|^p d\lambda\right)^{\frac{1}{p}} \leq \left(\int_{\mathbb{R}} |f|^q d\lambda\right)^{\frac{1}{q}} ?$$

(Also, explain whether or not we can use (1b) to answer this.)

(b) Let $g\colon \Omega \to \mathbb{R}$ a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then for all $\lambda > 0$ it holds that

$$\mathbb{P}(|g| \geq \lambda) \leq e^{-\lambda} \mathbb{E} e^{|g|} ?$$

(c) Let $f\colon \Omega \to \mathbb{R}$ be an integrable random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then the variance $\mathrm{Var}(f) := \mathbb{E}[(f - \mathbb{E}(f))^2]$ is always finite?

(7) **convergence in probability**
For $n \geq 1$ let $f_n, g_n, f, g \in \mathcal{L}_0(\Omega, \mathcal{F}, \mathbb{P})$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. Show the following:

(a) If $f_n \xrightarrow{\mathbb{P}} f$, then $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence in probability.

(b) Uniqueness of the limit: if $f_n \xrightarrow{\mathbb{P}} f$ and $f_n \xrightarrow{\mathbb{P}} g$, then $\mathbb{P}(f = g) = 1$.

(c) If $f_n \xrightarrow{\mathbb{P}} f$ and $g_n \xrightarrow{\mathbb{P}} g$ and $\lambda, \mu \in \mathbb{R}$, then

$$\lambda f_n + \mu g_n \xrightarrow{\mathbb{P}} \lambda f + \mu g \text{ and } f_n g_n \xrightarrow{\mathbb{P}} fg.$$

(**Hint:** you may use Theorem 9.2.4. (4))