Probability Theory 2 – Amlan Banaji spring term 2026
Exercises 3 Solutions, 10:00-11:30 MaD 381 April 13, 2026

(1) **Continuous mapping theorem**

Consider an arbitrary subsequence $n_1 < n_2 < \cdots$. Since $f_{n_k} \xrightarrow{\mathbb{P}} f$, by Theorem 9.2.4 (4) there is a further subsequence with $f_{n_{k_l}} \xrightarrow{a.s.} f$. Since φ is continuous, if $f_{n_{k_l}}(\omega) \to f(\omega)$ then $\varphi(f_{n_{k_l}}(\omega)) \to \varphi(f(\omega))$, so $\varphi(f_{n_{k_l}}) \xrightarrow{a.s.} \varphi(f)$. Therefore by Theorem 9.2.4 (4) again, $\varphi(f_n) \xrightarrow{\mathbb{P}} \varphi(f)$.

(2) **Sequence of coin tosses**

The probability is 0. Indeed, consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = \{H, T\}$, $\mathcal{F} = 2^{\{H,T\}}$, and $\mathbb{P}(\{H\}) = \mathbb{P}(\{T\}) = 1/2$ (this represents one toss of a fair coin). A sequence of independent tosses of a fair coin is represented by the infinite product probability space $(\bigtimes_{n=1}^{\infty}\Omega, \otimes_{n=1}^{\infty}\mathcal{F}, \otimes_{n=1}^{\infty}\mathbb{P})$. For $n \in \mathbb{N}$ define $X_n \colon \bigtimes_{n=1}^{\infty}\Omega \to \mathbb{R}$ by

$$X_n((\omega_1, \omega_2, \dots)) = \begin{cases} 1 & \text{if } \omega_n = H, \\ -1 & \text{if } \omega_n = T. \end{cases}$$

Then $X_1, X_2, \dots$ are i.i.d. random variables with $\mathbb{E}(X_n) = 0$ and $\mathbb{E}(X_n^4) = 1 < \infty$. Therefore by the Strong Law of Large Numbers,

$$\otimes_{n=1}^{\infty}\mathbb{P}\left(\frac{X_1 + \cdots + X_n}{n} \xrightarrow[n\to\infty]{} 0\right) = 1.$$

But if $\frac{X_1+\cdots+X_n}{n} \xrightarrow[n\to\infty]{} 0$ then there exists $N \in \mathbb{N}$ such that for all $n \geq N$ it holds that $\frac{X_1+\cdots+X_n}{n} < 0.001$, so there are only finitely many n_k such that at least 51% of the first n_k coin tosses are Heads.

(3) **convergence?**

(a) $f_n(x) \to \mathbf{1}_{\{1\}}(x)$ for all $x \in [0,1]$. So $f_n \xrightarrow{\lambda-a.s.} 0$ and $f_n \xrightarrow{\delta_1-a.s.} 1$. Since $|f_n| \leq 1$ for all n, by the dominated convergence theorem, $\mathbb{E}_\mu \lim_n f_n = \lim_n \mathbb{E}_\mu f_n$ for both λ and δ_1.

(b) $f_n(x) \to \infty\mathbf{1}_{\{1\}}(x)$ for all $x \in [0,1]$. Indeed, for $x \in [0,1)$, $nx^n \to 0$ because "exponential beats polynomial." So $f_n \xrightarrow{\lambda-a.s.} 0$ and $f_n \xrightarrow{\delta_1-a.s.} \infty$. Now

$$\mathbb{E}_{\delta_1}\lim_n f_n = \infty = \lim_n n = \lim_n \mathbb{E}_{\delta_1} f_n.$$

But

$$\mathbb{E}_\lambda(f_n) = \int_0^1 nx^n dx = \frac{n}{n+1}[x^{n+1}]_0^1 = \frac{n}{n+1} \to 1,$$

while $\mathbb{E}_\lambda(\lim_n f_n) = 0$.

(c) $f_n(x) \to \infty\mathbf{1}_{\{0\}}(x)$ for all $x \in [0,1]$. So $f_n \to 0$ a.s. for both λ and δ_0. So for all n, $0 = \mathbb{E}_{\delta_1} f_n = \mathbb{E}_{\delta_1}\lim_n f_n$. Now, $\mathbb{E}_\lambda f_n = \frac{4^n}{2^n} = 2^n \to \infty$, but $\mathbb{E}_\lambda(\lim_n f_n) = 0$.

(4) **convergence in probability: equivalences**

For the forward implication, since $x \mapsto \min\{1, x\}$ is continuous for all $p \in (0,\infty)$, by the continuous mapping theorem $f_n \xrightarrow{\mathbb{P}} f$ implies $\min\{1, |f_n - f|^p\} \xrightarrow{\mathbb{P}} 0$. Moreover $\sup_n(\min\{1, |f_n - f|^p\}) \leq 1$ so $\mathbb{E}(\sup_n(\min\{1, |f_n - f|^p\})) \leq 1 < \infty$. Now Proposition 9.3.3 (4) implies $\mathbb{E}(\min\{1, |f_n - f|\}) \to 0$.

For the backward implication, let $p \in (0,\infty)$ be such that $\mathbb{E}(\min\{1, |f_n - f|^p\}) \to 0$, and let $0 < \varepsilon \leq 1$. Then by Markov's inequality,

$$\mathbb{P}(|f_n - f| \geq \varepsilon) = \mathbb{P}(|f_n - f|^p \geq \varepsilon^p) = \mathbb{P}(\min\{1, |f_n - f|^p\} \geq \varepsilon^p) \leq \varepsilon^{-n}\mathbb{E}(\min\{1, |f_n - f|^p\}) \to 0$$

by assumption. The $\varepsilon > 1$ case follows trivially since $\mathbb{P}(|f_n - f| > \varepsilon) \leq \mathbb{P}(|f_n - f|^p \geq \min\{\varepsilon, 1\})$.

(5) $\mathcal{L}_0(\Omega, \mathcal{F}, \mathbb{P})$ **a metric space?**

(a) is true: $D(f,g) = 0 \Leftrightarrow \min\{1, |f-g|\} = 0\, a.s. \Leftrightarrow |f-g| = 0\, a.s. \Leftrightarrow f = g\, a.s.$

(b) is clearly true.

(c) is also true. Indeed, for $a, b \geq 0$, $\min\{1,a\} + \min\{1,b\} = a+b = \min\{1, a+b\}$ if $a+b \leq 1$, while $\min\{1,a\} + \min\{1,b\} \geq 1 = \min\{1, a+b\}$ if $a+b > 1$. Therefore by the definition of D,

$$D(f,h) \leq \mathbb{E}(\min\{1, |f-g| + |g-h|\}) \leq D(f,g) + D(g,h).$$

Remark: $\mathcal{L}_0(\Omega, \mathcal{F}, \mathbb{P})$ turns into a metric space if we identify functions which are a.s. equal, i.e. the space of equivalence classes for the equivalence relation $f \sim g$ if $f = g$ a.e. becomes a metric space.

(6) **uniform integrability**

(a) First note that if $a, b \in \mathbb{R}$ and $|a+b| \geq c$ then either $|a| \geq c/2$ or $|b| \geq c/2$ (or both), and $|a+b| \leq 2\max\{|a|, |b|\}$. Therefore

$$\mathbf{1}_{|f+g| \geq c/2} |f+g| \leq 2[\mathbf{1}_{|f| \geq c/2}|f| + \mathbf{1}_{|g| \geq c/2}|g|].$$

Integrating gives the desired inequality.

(b) Using part (a), for all $c > 0$,

$$\begin{aligned} \sup_i \mathbb{E}(\mathbf{1}_{|\alpha f_i + \beta g_i| \geq c} |f+g|) &\leq 2 \sup_i (\mathbb{E}(\mathbf{1}_{|\alpha f_i| \geq c/2} |\alpha f_i|) + \mathbb{E}(\mathbf{1}_{|\beta g_i| \geq c/2} |\beta g_i|)) \\ &\leq 2(\alpha \sup_i \mathbb{E}(\mathbf{1}_{|f_i| \geq c/(2|\alpha|)} |f_i| + \beta \sup_j \mathbb{E}(\mathbf{1}_{|g_j| \geq c/(2|\beta|)} |g_j|)) \xrightarrow[c \to \infty]{} 0, \end{aligned}$$

where we used the obvious convention that if α or β is 0 then we ignore the corresponding terms.

(c) We observed in lectures that by the dominated convergence theorem, for each i, $\{f_i\}$ is u.i. Therefore for all $\varepsilon > 0$ we can find $c_1, \ldots, c_N > 0$ such that for each i, $\mathbb{E}(\mathbf{1}_{|f_i| \geq c_i} |f_i|) \leq \varepsilon$. Therefore if $c := \max\{c_1, \ldots, c_N\}$ then

$$\max_{1 \leq i \leq N} \mathbb{E}(\mathbf{1}_{|f_i| \geq c} |f_i|) \leq \max_{1 \leq i \leq N} \mathbb{E}(\mathbf{1}_{|f_i| \geq c_i} |f_i|) \leq \varepsilon,$$

so $\{f_1, \ldots, f_N\}$ is u.i.