Please hand in your answer to Question (1)* by email before the start of the exercise session, or on paper at the start of the exercise session.
You do not need to hand in the rest of your solutions (unless you are unable to attend the exercise session), but be prepared to present one or two of the ones you have completed (I will choose which) during the exercise session. If you are unable to attend the exercise session, please email me all the questions you have attempted by 10am on the day of the exercise session.

(1) * **Continuous mapping theorem**
Assume that $f, f_1, f_2, \dots \in \mathcal{L}_0(\Omega, \mathcal{F}, \mathbb{P})$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. If $\varphi\colon \mathbb{R} \to \mathbb{R}$ is continuous and $f_n \xrightarrow{\mathbb{P}} f$, show that $\varphi(f_n) \xrightarrow{\mathbb{P}} \varphi(f)$.
(**Hint:** use Theorem 9.2.4. (4))

(2) **Sequence of coin tosses**
Consider an infinite sequence of independent tosses of a fair coin. What is the probability that there exists a subsequence of natural numbers $1 \le n_1 \le n_2 \le \cdots$ such that for all $k \in \mathbb{N}$, at least 51% of the first n_k coin tosses are Heads? Explain your answer.

(3) **convergence?**
Consider the probability space $([0,1], \mathcal{B}([0,1]), \mu)$ (for some measure μ), and consider the following sequences $(f_n)_n$ given by
(a) $x \mapsto x^n$, $n \in \mathbb{N}$
(b) $x \mapsto nx^n$, $n \in \mathbb{N}$
(c) $x \mapsto 4^n \mathbb{1}_{[0,1/2^n)}$, $n \in \mathbb{N}$.
Find out for $\mu = \lambda$ (Lebesgue measure) and for $\mu = \delta_1$ whether the sequences converge a.s. Which are the exception sets, where the sequences do not converge? For which of the sequences and which measure do we have $\mathbb{E}_\mu \lim_n f_n = \lim_n \mathbb{E}_\mu f_n$?

(4) **convergence in probability: equivalences**
For $n \ge 1$ let $f, f_n \in \mathcal{L}_0(\Omega, \mathcal{F}, \mathbb{P})$. Show that $f_n \xrightarrow{\mathbb{P}} f$ if and only if there is a $p \in (0, \infty)$ for which $\mathbb{E}(\min\{1, |f_n - f|^p\}) \to 0$.
Hint: Use Proposition 9.3.3 (4) and Markov's inequality.

(5) $\mathcal{L}_0(\Omega, \mathcal{F}, \mathbb{P})$ **a metric space?**
For $f, g \in \mathcal{L}_0(\Omega, \mathcal{F}, \mathbb{P})$ define $D(f,g) := \mathbb{E}(\min\{1, |f - g|\})$. Check whether the following assertions are true:
(a) $D(f,g) = 0 \iff P(f = g) = 1$,
(b) $D(f,g) = D(g,f)$,
(c) $D(f,h) \le D(f,g) + D(g,h)$ for $f, g, h \in \mathcal{L}_0(\Omega, \mathcal{F}, \mathbb{P})$.

(6) **uniform integrability**
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.
(a) For $c > 0$ and random variables $f, g\colon \Omega \to \mathbb{R}$, prove that
$$\mathbb{E}(\mathbf{1}_{|f+g| \ge c} |f + g|) \le 2[\mathbb{E}(\mathbf{1}_{|f| \ge c/2} |f|) + \mathbb{E}(\mathbf{1}_{|g| \ge c/2} |g|)].$$
(b) Let $f_1, f_2, \dots$ and $g_1, g_2, \dots$ be random variables $\Omega \to \mathbb{R}$, assume $(f_n)_{n=1}^\infty$ is u.i. and $(g_n)_{n=1}^\infty$ is u.i., and let $\alpha, \beta \in \mathbb{R}$. Prove that $(\alpha f_n + \beta g_n)_{n=1}^\infty$ is u.i.
(c) If $f_1, \dots, f_N \in \mathcal{L}_1(\Omega, \mathcal{F}, \mathbb{P})$, prove that $\{f_1, \dots, f_N\}$ is u.i.