PROBABILITY THEORY 2 – AMLAN BANAJI spring term 2026
EXERCISES 4 SOLUTIONS, 10:00-11:30 MAD 381 April 20, 2026

(1) * **Hahn and Jordan**

Let $f(y) = \sin(2\pi y)$ for $0 \le y \le 2$ (drawing the graph of y is a good idea). Let

$$\Omega^+ := [0,1/2) \cup [1,3/2), \qquad \Omega^- := [1/2,1) \cup [3/2,2].$$

Then $\max\{f,0\} = \sin(2\pi y)\mathbf{1}_{\Omega^+}$ and $\max\{-f,0\} = -\sin(2\pi y)\mathbf{1}_{\Omega^-}$. Now $\Omega = \Omega^+ \cup \Omega^-$ is a Hahn decomposition of Ω, and the corresponding Hahn–Jordan decomposition is

$$\nu_J^\pm = \int_{[0,2]\cap A} f^\pm(y)\,d\lambda(y).$$

(2) **Cantor set**

Each C_n is a finite union of 2^n closed intervals, so each $C_n \in \mathcal{B}([0,1])$. Therefore

$$C := \bigcap_{n=1}^{\infty} C_n \in \mathcal{B}([0,1]).$$

Moreover, by induction, $\lambda(C_n) = (2/3)^n$ for all n. Also $C_0 \supset C_1 \supset C_2 \supset \cdots$, so by continuity from above,

$$\lambda(C) = \lambda\left(\bigcap_{n=1}^{\infty} C_n\right) = \lim_{n\to\infty} \lambda(C_n) = \lim_{n\to\infty}\left(\frac{2}{3}\right)^n = 0.$$

(3) **monotone function is measurable**

Recall from Proposition 3.3.6 (3) that the intervals of the form $(-\infty,b)$ for $b \in \mathbb{R}$ generate the Borel σ-algebra on $\mathbb{R}$, so by Lemma 3.3.17 (3) the intervals of the form $(-\infty,b)\cap[0,\infty)$ generate $\mathcal{B}([0,\infty))$. If $b < 0$ then $G^{-1}((-\infty,b)\cap[0,\infty)) = G^{-1}(\varnothing) = \varnothing$. If $b \ge 0$ and $y \in G^{-1}((-\infty,b)\cap[0,\infty))$ then for all $0 \le x \le y$ we have $G(x) \le G(y)$ so $y \in G^{-1}((-\infty,b)\cap[0,\infty))$, thus $G^{-1}((-\infty,b)\cap[0,\infty))$ is of the form $[0,z]$ or $[0,z)$. In either case, $G^{-1}((-\infty,b)\cap[0,\infty)) \in \mathcal{B}(([0,\infty))$. So G is $(\mathcal{B}([0,\infty)),\mathcal{B}(\mathbb{R}))$-measurable.

(4) **a.s. convergence**

Yes. Indeed, for $k \in \mathbb{N}$ let

$$\Omega_0 := \{\omega \in \Omega : f_k(\omega) \xrightarrow[k\to\infty]{} f(\omega)\}, \qquad \Omega_k := \{\omega \in \Omega : g_k(\omega) = f_k(\omega)\}.$$

By our assumptions, $\mathbb{P}(\Omega_k) = 1$ for all $k \ge 0$. Now

$$\mathbb{P}\left(\bigcap_{k=0}^{\infty}\Omega_k\right) = 1 - \mathbb{P}\left(\Omega \setminus \left(\bigcap_{k=0}^{\infty}\Omega_k\right)\right) = 1 - \mathbb{P}\left(\bigcup_{n=1}^{\infty}(\Omega\setminus\Omega_k)\right) = 1 - 0 = 1.$$

But if $\omega \in \bigcap_{k=0}^{\infty}\Omega_k$ then $\lim_{k\to\infty} g_k(\omega) = \lim_{k\to\infty} f_k(\omega) = f(\omega)$.

(5) **uniform integrability**

(a) $(f_k)_{k=1}^{\infty}$ is u.i. by Lemma 9.4.4, because for $G(y) = y^2$ we have

$$\sup_{k\ge1}\mathbb{E}(G(|f_k|)) = \sup_{k\ge1}(\mathbb{E}|f_k|^2) = 1 < \infty.$$

(b) $(g_k)_{k=1}^{\infty}$ is not generally u.i. For example, let g_k be the constant function $k^{1/4}$ on Ω. Then

$$\mathbb{E}(g_k) = k^{1/4} \xrightarrow[k\to\infty]{} \infty.$$

Therefore by the contrapositive of Lemma 9.4.3, $(g_k)_{k=1}^{\infty}$ is not u.i.

(c) $(h_k)_{k=1}^{\infty}$ is u.i., again by applying Lemma 9.4.4 with $G(y) = y^2$. Indeed, for all k,

$$\mathbb{E}(G(|h_k|)) = \mathbb{E}(h_k^2) = \mathbb{E}\left(\frac{(X_1 + \cdots + X_k)^2}{k}\right) = \frac{1}{k}\mathrm{Var}(X_1 + \cdots + X_k) = \frac{1}{k}(\mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_k)) = \frac{k}{k} = 1.$$

We used that $\mathbb{E}(X_1 + \cdots + X_k) = 0 + \cdots + 0 = 0$ by linearity, so $\mathbb{E}(X_1 + \cdots + X_k)^2 = \mathrm{Var}(X_1 + \cdots + X_k)$.

(6) **integrable bound $\Longrightarrow$ u.i.**

For $\Longrightarrow$ we have

$$\sup_{n\in\mathbb{N}} \mathbb{E}(|f_n|\mathbf{1}_{|f_n|\ge c}) \le \sup_{n\in\mathbb{N}} \mathbb{E}(|f_n|\mathbf{1}_{\sup_k |f_k|\ge c}) \le \mathbb{E}(\sup_m |f_m|\mathbf{1}_{\sup_k |f_k|\ge c}) \xrightarrow[c\nearrow\infty]{} 0,$$

The last limit is by the short argument from the start of Section 9.4 using the dominated convergence theorem, since $\sup_m |f_m|$ is assumed to be an integrable random variable.

For $\not\Longleftarrow$, let the f_n be i.i.d. standard normal random variables. We know from Corollary 8.10.2 that each f_n has finite variance and in particular is integrable. Therefore

$$\sup_{n\in\mathbb{N}} \mathbb{E}(\mathbf{1}_{|f_n|\ge c}|f_n|) = \mathbb{E}(\mathbf{1}_{|f_1|\ge c}|f_1|) \xrightarrow[c\nearrow\infty]{} 0.$$

Therefore $(f_n)_{n=1}^{\infty}$ is u.i. But for all $k, n \in \mathbb{N}$, $\mathbb{P}(f_n > k) > 0$, so by independence,

$$\mathbb{P}(\max\{f_1, \ldots, f_n\} \le k) = (1 - \mathbb{P}(f_n > k))^n \xrightarrow[n\to\infty]{} 0.$$

By continuity from above, we see that $\mathbb{P}(f_n \le k \forall n \in \mathbb{N}) = 0$. So by countable subadditivity,

$$\mathbb{P}(\sup_n |f_n| < \infty) = \mathbb{P}\left(\bigcup_{k=1}^{\infty}\{f_n \le k \forall n \in \mathbb{N}\}\right) = \sum_{k=1}^{\infty} 0 = 0.$$

In particular, $\mathbb{E}(\sup_n |f_n|) = \infty$.

(7) **absolutely continuous measures**

For the forward implication, our assumption is that $\mathbb{P}_1(A) = 0$ iff $\mathbb{P}(A) = 0$. Now, $\mathbb{P}(A) = 0$ iff $A \cap \{\omega_1, \ldots, \omega_k\} = \varnothing$. Therefore $\mathbb{P}_1$ satisfies $\mathbb{P}_1(\Omega \setminus \{\omega_1, \ldots, \omega_k\}) = 0$, so $\mathbb{P}_1(\{\omega_1, \ldots, \omega_k\}) = 1$. Therefore $\mathbb{P}_1 = \sum_{k=1}^{n} a_k \delta_{\omega_k}$ for some $a_1, \ldots, a_n \in [0, 1]$ with $\sum_{k=1}^{n} a_k = 1$. But since $\mathbb{P}(\{\omega_k\}) > 0$ for all k, we must have $\mathbb{P}_1(\{\omega_k\}) = a_k > 0$ for all k.

For the backward implication, note that the assumptions on $\{a_1, \ldots, a_n\}$ mean that $\mathbb{P}_1 = \sum_{k=1}^{n} a_k \delta_{\omega_k}$ is indeed a probability measure. Then $\mathbb{P}(A) = 0$ iff $A \cap \{\omega_1, \ldots, \omega_k\} = \varnothing$ iff $\mathbb{P}_1(A) = 0$. So $\mathbb{P} \ll \mathbb{P}_1$ and $\mathbb{P}_1 \ll \mathbb{P}$.