Please hand in your answer to Question (1)* by email before the start of the exercise session, or on paper at the start of the exercise session.
You do not need to hand in the rest of your solutions (unless you are unable to attend the exercise session), but be prepared to present one or two of the ones you have completed (I will choose which) during the exercise session. If you are unable to attend the exercise session, please email me all the questions you have attempted by 10am on the day of the exercise session.

(1) * **Hahn and Jordan**
Let ν be the measure on $([0,2], \mathcal{B}([0,2]))$ given by

$$\nu([0,x]) := \int_0^x \sin(2\pi y)dy, \quad x \in [0,2].$$

Find a Hahn decomposition and the corresponding Hahn–Jordan decomposition for ν.

(2) **Cantor set**
Let $C \subseteq [0,1]$ denote the middle-third Cantor set, which can be generated as follows:
(a) $C_0 = [0,1]$
(b) C_{n+1} is produced from C_n by removing from each interval the middle third part (as an open interval) so that one gets

$$\begin{aligned} C_1 &= [0,1/3] \cup [2/3,1], \\ C_2 &= [0,1/9] \cup [2/9,3/9] \cup [6/9,7/9] \cup [8/9,1], \end{aligned}$$

and so on.
(c) $C := \bigcap_{n\in\mathbb{N}} C_n$.
Argue why we have $C \in \mathcal{B}([0,1])$, and compute $\lambda(C)$, where λ denotes the Lebesgue measure.

(3) **monotone function is measurable**
Let $G\colon [0,\infty) \to [0,\infty)$ be a nondecreasing function (i.e. if $0 \le x \le y$ then $G(x) \le G(y)$). Show that G is $(\mathcal{B}([0,\infty)), \mathcal{B}(\mathbb{R}))$-measurable.
Hint: you may use Proposition 3.3.6 and Lemma 3.3.17.

(4) **a.s. convergence**
Assume that $(f_k)_{k\in\mathbb{N}}, (g_k)_{k\in\mathbb{N}} \subseteq \mathcal{L}_0(\Omega,\mathcal{F},\mathbb{P})$ and $f_k \xrightarrow{a.s.} f$ where $f \in \mathcal{L}_0(\Omega,\mathcal{F},\mathbb{P})$. If it holds that

$$\mathbb{P}(g_k = f_k) = 1, \quad \text{for all } k \in \mathbb{N},$$

does it then follow that $g_k \xrightarrow{a.s.} f$?

(5) **uniform integrability**
For which sequences can one prove that they are u.i. (uniformly integrable)?
(a) $(f_k)_{k\in\mathbb{N}} \subseteq \mathcal{L}_0(\Omega,\mathcal{F},\mathbb{P})$ with $\mathbb{E}|f_k|^2 = 1$
(b) $(g_k)_{k\in\mathbb{N}} \subseteq \mathcal{L}_0(\Omega,\mathcal{F},\mathbb{P})$ with $\mathbb{E}|g_k|^4 = k$
(c) $(h_k)_{k\in\mathbb{N}}$ given by $h_k = \frac{|X_1+\cdots+X_k|}{\sqrt{k}}$, where $(X_k)_{k\in\mathbb{N}}$ is a sequence of i.i.d. standard normal random variables

(6) **integrable bound $\Longrightarrow$ u.i.**
We know that if $(f_n)_{n=1}^\infty \subseteq \mathcal{L}_0(\Omega,\mathcal{F},\mathbb{P})$ is u.i or if $\mathbb{E}(\sup_n |f_n|) < \infty$, then $f_n \xrightarrow{\mathbb{P}} f$ implies $f_n \xrightarrow{L_1} f$. Show that

$$\mathbb{E}(\sup_n |f_n|) < \infty \implies (f_n)_{n=1}^\infty \ u.i.$$

but

$$(f_n)_{n=1}^{\infty}\ u.i. \not\Longrightarrow \mathbb{E}(\sup_n |f_n|) < \infty.$$

Hint: There are many different possible approaches to $\not\Longrightarrow$, but one is to let the f_n be i.i.d. normal random variables.

(7) **absolutely continuous measures**
Assume that $n \in \mathbb{N}$ and $(\Omega, 2^{\Omega}, \mathbb{P})$ is a probability space where $\mathbb{P} = \frac{1}{n}\sum_{k=1}^{n} \delta_{\omega_k}$ with distinct points $\omega_1, \ldots, \omega_n \in \Omega$. Show that if $\mathbb{P}_1$ is a probability measure on $(\Omega, 2^{\Omega})$, then

$$(\mathbb{P}_1 \ll \mathbb{P} \text{ and } \mathbb{P} \ll \mathbb{P}_1) \iff \exists a_1, \ldots, a_n \in (0,1] \text{ with } a_1 + \cdots + a_n = 1 \text{ such that } \mathbb{P}_1 = \sum_{k=1}^{n} a_k \delta_{\omega_k}.$$