

(1) * **conditional monotone convergence**

As a pointwise limit of \mathcal{G} -measurable functions, $\lim_{n \rightarrow \infty} \mathbb{E}[f_n | \mathcal{G}]$ is \mathcal{G} -measurable by Remark 9.1.2. Let $B \in \mathcal{G}$. Since $(f_n)_n$ is pointwise increasing, so is $(\mathbb{E}[f_n | \mathcal{G}])_n$, and so is $(\mathbb{E}[f_n | \mathcal{G}] \mathbf{1}_B)_n$. Therefore we can apply the monotone convergence theorem twice to get

$$\mathbb{E}((\lim_{n \rightarrow \infty} \mathbb{E}[f_n | \mathcal{G}]) \mathbf{1}_B) = \lim_{n \rightarrow \infty} \mathbb{E}(\mathbb{E}[f_n | \mathcal{G}] \mathbf{1}_B) = \lim_{n \rightarrow \infty} \mathbb{E}(f_n \mathbf{1}_B) = \mathbb{E}(f \mathbf{1}_B) = \mathbb{E}(\mathbb{E}[f | \mathcal{G}] \mathbf{1}_B).$$

(2) **conditional expectation properties**

(a) **conditional expectation is linear:**

By the definition of $\mathbb{E}[\cdot | \mathcal{G}]$, we have that $\mathbb{E}[af + bg | \mathcal{G}]$ and $a\mathbb{E}[f | \mathcal{G}] + b\mathbb{E}[g | \mathcal{G}]$ are \mathcal{G} -measurable and integrable. For all $G \in \mathcal{G}$ we have

$$\mathbb{E}((af + bg) \mathbf{1}_G) = a\mathbb{E}(f \mathbf{1}_G) + b\mathbb{E}(g \mathbf{1}_G) = a\mathbb{E}(\mathbb{E}[f | \mathcal{G}] \mathbf{1}_G) + b\mathbb{E}(\mathbb{E}[g | \mathcal{G}] \mathbf{1}_G) = \mathbb{E}((a\mathbb{E}[f | \mathcal{G}] + b\mathbb{E}[g | \mathcal{G}]) \mathbf{1}_G).$$

The result follows by the definition of conditional expectation.

(b) **conditional expectation with independent condition:**

For all $G \in \mathcal{G}$ we have

$$\mathbb{E}(\mathbb{E}[f | \mathcal{G}] \mathbf{1}_G) = \mathbb{E}(f \mathbf{1}_G) = (\mathbb{E}f)(\mathbb{E} \mathbf{1}_G) = \mathbb{E}((\mathbb{E}f) \mathbf{1}_G),$$

and the result follows.

(3) **conditional expectation: partitions**

(a) We have $\mathbb{E}[\mathbf{1}_B | \mathcal{G}] = a\mathbf{1}_A + b\mathbf{1}_{A^c}$ a.s., for some $a, b \in \mathbb{R}$, since this is the general form of a \mathcal{G} -measurable function. Then

$$\mathbb{P}(B \cap A) = \mathbb{E}(\mathbb{E}[\mathbf{1}_B | \mathcal{G}] \mathbf{1}_A) = \mathbb{E}(a\mathbf{1}_A + b\mathbf{1}_{A^c}) = a\mathbb{P}(A).$$

So $a = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} =: \mathbb{P}(B|A)$. Similarly $b = \mathbb{P}(B|A^c)$.

(b) Any \mathcal{G} -measurable function can be represented as $\sum_{k=1}^{\infty} a_k \mathbf{1}_{A_k}$ for some real numbers a_1, a_2, \dots , so this is the form of $\mathbb{E}[f | \mathcal{G}]$. Now

$$\mathbb{E}(f \mathbf{1}_{A_m}) = \mathbb{E}(\mathbb{E}[f | \mathcal{G}] \mathbf{1}_{A_m}) = \mathbb{E}\left(\left(\sum_{k=1}^{\infty} a_k \mathbf{1}_{A_k}\right) \mathbf{1}_{A_m}\right) = a_m \mathbb{P}(A_m).$$

Therefore $a_m = \frac{\mathbb{E}(f \mathbf{1}_{A_m})}{\mathbb{P}(A_m)}$.

If $\mathbb{P}(A_1) = 0$ then $\sum_{k=1}^{\infty} a_k \mathbf{1}_{A_k} = \sum_{k=2}^{\infty} a_k \mathbf{1}_{A_k}$ so

$$\mathbb{E}[f | \mathcal{G}] = \sum_{k=2}^{\infty} \frac{\mathbb{E}(f \mathbf{1}_{A_k})}{\mathbb{P}(A_k)} \mathbf{1}_{A_k} \quad \text{a.s.}$$

(4) **conditional expectation calculations**

(a) We have

$$\sigma(X) = \{A \in \mathcal{B}([0, 1]) : x \in A \Leftrightarrow 1 - x \in A\}, \quad \sigma(Y) = \sigma([0, 1/2], (1/2, 1]).$$

By Problem (3b) we get

$$\begin{aligned}\mathbb{E}[X|\sigma(Y)] &= \frac{\mathbb{E}(X\mathbf{1}_{[0,1/2]})}{\lambda([0,1/2])}\mathbf{1}_{[0,1/2]} + \frac{\mathbb{E}(X\mathbf{1}_{(1/2,1]})}{\lambda((1/2,1])}\mathbf{1}_{(1/2,1]} = 2 \int_0^{1/2} (1/2-x)dx\mathbf{1}_{[0,1/2]} + 2 \int_{1/2}^1 (x-1/2)\mathbf{1}_{(1/2,1]} \\ &= \frac{1}{4}\mathbf{1}_{[0,1/2]} + \frac{1}{4}\mathbf{1}_{(1/2,1]} = 1/4.\end{aligned}$$

To get $\mathbb{E}[Y|\sigma(X)]$ we see that for all $A \in \sigma(X)$,

$$\int_A Yd\lambda = \lambda(A \cap [0, 1/2]) = \lambda(A \cap (1/2, 1]) = \lambda(A)/2 = \int_A 1/2d\lambda.$$

Since $\mathbb{E}[Y|\sigma(X)]$ is a.s. unique, we get $\mathbb{E}[Y|\sigma(X)] = 1/2$ a.s.

(b) Now

$$\sigma(X) = \mathcal{B}([0, 1]), \quad \sigma(Y) = \sigma((0, 1/8], (1/8, 2/8], \dots, (7/8, 1]).$$

Now Y is $\mathcal{B}([0, 1])$ -measurable so $\mathbb{E}[Y|\sigma(X)] = Y$. By Problem (3b) we have $\mathbb{E}[X|\sigma(Y)] = \sum_{k=1}^8 a_k \mathbf{1}_{(t_{k-1}, t_k]}$ a.s., where

$$a_k = \frac{\mathbb{E}(X\mathbf{1}_{(t_{k-1}, t_k]})}{\lambda((t_{k-1}, t_k])} = \frac{\int_{t_{k-1}}^{t_k} xdx}{t_k - t_{k-1}} = \frac{t_k^2 - t_{k-1}^2}{2(t_k - t_{k-1})} = \frac{t_k + t_{k-1}}{2}.$$

(5) complex numbers

(a)

$$\frac{1}{(1-i)^3} = \frac{1}{(1-2i+i^2)(1-i)} = \frac{1}{2i(i-1)} = \frac{1}{2(-1-i)} = \frac{-(i-1)}{2(i+1)(i-1)} = \frac{i-1}{4}.$$

(b)

$$\overline{e^{a+ib}} = \overline{e^a(\cos b + i \sin b)} = e^a(\cos b - i \sin b) = e^a(\cos(-b) + i \sin(-b)) = e^{a-ib}.$$

(c) Without loss of generality we can assume that $\alpha, \beta \in [0, 2\pi)$. Now the inequality boils down to the fact that the Euclidean distance between two points on the unit circle is at most the length of the arc between them.

(6) change of variable

(a) We take $([0, 1]^3, \mathcal{B}([0, 1]^3), \mathbb{P})$ where \mathbb{P} is the restriction of 3-dimensional Lebesgue measure to $[0, 1]^3$, and take $X((x, y, z)) = x$, $Y((x, y, z)) = y$, $Z((x, y, z)) = z$. The region $X + Y + Z \leq 1$ is a solid tetrahedron with vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. Then by Tonelli's theorem,

$$\begin{aligned}\mathbb{P}(X + Y + Z \leq 1) &= \mathbb{P}(x + y + z \leq 1) = \int_0^1 \int_0^1 \int_0^1 \mathbf{1}_{\{x+y+z \leq 1\}}((x, y, z)) dx dy dz \\ &= \int_0^1 \int_0^{1-z} \int_0^{1-y-z} \mathbf{1}_{\{1-y-z \geq 0\}}(x) dx dy dz \\ &= \int_0^1 \int_0^{1-z} (1-y-z) dy dz = \int_0^1 \left((1-z)^2 - \frac{(1-z)^2}{2} \right) dy dz \\ &= -\frac{1}{2} \frac{1}{3} [(1-z)^3]_0^1 = \frac{1}{3!}.\end{aligned}$$

(b) For all $t \in \mathbb{R}$,

$$\mathbb{P}(F^{-1}(Y) \leq t) = \mathbb{P}(Y \leq F(t)) = \lambda(\{s \in [0, 1] : s \leq F(t)\}) = \lambda([0, F(t)]) = F(t) = \mathbb{P}(A \leq t),$$

which is what we needed to show.