Please hand in your answer to Question (1)* by email before the start of the exercise session, or on paper at the start of the exercise session.
You do not need to hand in the rest of your solutions (unless you are unable to attend the exercise session), but be prepared to present one or two of the ones you have completed (I will choose which) during the exercise session. If you are unable to attend the exercise session, please email me all the questions you have attempted by 10am on the day of the exercise session.

(1) * **conditional monotone convergence**
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-algebra, and assume that f is an integrable random variable with $f \geq 0$ a.s. Use the monotone convergence theorem (Theorem 8.2.2) to show that for random variables with $0 \leq f_n \nearrow f$ a.s. as $n \to \infty$, it holds that

$$\mathbb{E}[f_n|\mathcal{G}] \xrightarrow[n\to\infty]{} \mathbb{E}[f|\mathcal{G}] \quad \text{a.s.}$$

(2) **conditional expectation properties**
Let $f, g \in \mathcal{L}_1(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$.
(a) **conditional expectation is linear:**
Show that for $a, b \in \mathbb{R}$, it holds that

$$\mathbb{E}[af + bg|\mathcal{G}] = a\mathbb{E}[f|\mathcal{G}] + b\mathbb{E}[g|\mathcal{G}] \quad a.s.$$

(b) **conditional expectation with independent condition:**
We say that f is independent from $\mathcal{G}$ if

$$\mathbb{P}(\{f \in B\} \cap A) = \mathbb{P}(f \in B)\mathbb{P}(A), \quad \forall B \in \mathcal{B}(\mathbb{R}), A \in \mathcal{G}.$$

Show that if f is independent from $\mathcal{G}$ then

$$\mathbb{E}[f|\mathcal{G}] = \mathbb{E}f \quad a.s.$$

(3) **conditional expectation: partitions**
(a) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $A \in \mathcal{F}$ be such that $0 < \mathbb{P}(A) < 1$. For $\mathcal{G} = \{\Omega, \varnothing, A, A^c\}$ and $B \in \mathcal{F}$, find $\mathbb{E}[\mathbb{1}_B|\mathcal{G}]$. Use conditional probability.
(b) Let $f \in \mathcal{L}_1(\Omega, \mathcal{F}, \mathbb{P})$. Assume $\Omega = \cup_{k=1}^{\infty}\Omega_k$ where the sets $\Omega_1, \Omega_2, ... \in \mathcal{F}$ form a partition of Ω. Let $\mathbb{P}(\Omega_k) > 0$ for all $k \in \mathbb{N}$ and let $\mathcal{G} := \sigma(\Omega_1, \Omega_2, ...)$. Find the $a_k \in \mathbb{R}$ such that

$$\mathbb{E}[f|\mathcal{G}] = \sum_{k=1}^{\infty} a_k \mathbb{1}_{\Omega_k}.$$

What could one do if $\mathbb{P}(\Omega_1) = 0$?

(4) **conditional expectation calculations**
Let $([0,1], \mathcal{B}([0,1]), \lambda)$ be the probability space where $\mathcal{B}([0,1])$ denotes the Borel σ-algebra on $[0,1]$ and λ the Lebesgue measure. Write down expressions for the σ-algebras $\sigma(X)$ and $\sigma(Y)$ and then find the conditional expectations

$$\mathbb{E}[X|\sigma(Y)] \quad \text{and} \quad \mathbb{E}[Y|\sigma(X)]$$

if X and Y are given by
(a) $X(x) = \left|x - \frac{1}{2}\right|$ and $Y(x) = \mathbb{1}_{[0,0.5]}(x)$,
(b) $X(x) = x$ and $Y(x) = \sum_{k=1}^{8} t_k \mathbb{1}_{(t_{k-1}, t_k]}(x)$ where $t_k := \frac{k}{8}$.
(You do not need to justify the expressions for the σ-algebras, but you should justify the your answers for the conditional expectations.)

(5) **complex numbers**

(a) Simplify $\frac{1}{(1-i)^3}$.

(b) Let $\overline{z}$ denote the complex conjugate of a complex number z, and let $a, b \in \mathbb{R}$. Simplify $\overline{e^{a+ib}}$.

(c) Draw a picture which shows that $|e^{i\alpha} - e^{i\beta}| \le |\alpha - \beta|$ for all $\alpha, \beta \in \mathbb{R}$ (a formal calculation is not needed).

(6) **change of variable**

(a) Model a probability space where X, Y, Z are independent and uniform distributed on $[0, 1]$. Show that $\mathbb{P}(X + Y + Z \le 1) = \frac{1}{3!}$. What is the shape of the region described by $X + Y + Z \le 1$?

(b) Let A be a random variable with a strictly increasing distribution function F, so the inverse F^{-1} is well defined. Let Y be uniform distributed on $[0, 1]$. Explain why A and $F^{-1}(Y)$ have the same distribution.