Probability Theory 2 – Amlan Banaji spring term 2026
Exercises 6 Solutions, 10:00-11:30 MaD 381 May 4, 2026

(1) * **Fourier transforms of discrete measures**

(a) **Finitely supported measure:**

$$\widehat{\mu}(x) = \int_{\mathbb{R}^d} e^{i\langle x,y\rangle} d\mu = \sum_{k=1}^{n} \theta_k e^{i\langle x,a_k\rangle} = \sum_{k=1}^{n} \theta_k \Big(\cos(\langle x,a_k\rangle) + i\sin(\langle x,a_k\rangle)\Big).$$

(b) **Poisson distribution:** We have

$$\widehat{\mathrm{Pois}_\lambda}(x) = \sum_{k=0}^{\infty} e^{-\lambda} e^{ikx} \frac{\lambda^k}{k!} = e^{-\lambda}\frac{(e^{ix}\lambda)^k}{k!} = e^{\lambda(e^{ix}-1)}.$$

In the last step we used the series expansion for the exponential function. To justify the first equality rigorously (don't worry if you didn't do this in the hand-in), we can define for each $x \in \mathbb{R}$ and $n \in \mathbb{N}$ the functions $f_{x,n}, g_{x,n} : \mathbb{R} \to \mathbb{R}$ by $f_{x,n}(y) = \cos(xy)\mathbf{1}_{[-n,n]}(y)$ and $g_{x,n}(y) = \sin(xy)\mathbf{1}_{[-n,n]}(y)$. Then for each $y \in \mathbb{R}$ we have $f_{x,n}(y) \to \cos(xy)$ and $g_{x,n}(y) \to \sin(xy)$ as $n \to \infty$. Moreover, $|f_{x,n}(y)|, |g_{x,n}(y)| \leq 1$ for all x, n, y, and $\int_{\mathbb{R}} 1 d(\mathrm{Pois}_\lambda) < 1$. Therefore we can apply the dominated convergence theorem to get

$$\sum_{k=0}^{n} e^{-\lambda}\frac{\lambda^k}{k!}\cos(xk) = \int_{\mathbb{R}} f_{x,n} d(\mathrm{Pois}_\lambda) \xrightarrow[n\to\infty]{} \int_{\mathbb{R}} \cos(xy) d(\mathrm{Pois}_\lambda)(y),$$

so $\sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!}\cos(xk) = \int_{\mathbb{R}} \cos(xy) d(\mathrm{Pois}_\lambda)(y)$. Similarly, $\sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!}\sin(xk) = \int_{\mathbb{R}} \sin(xy) d(\mathrm{Pois}_\lambda)(y)$. The first equality now follows from the definition of the Fourier transform.

(2) **Fourier transform of uniform distribution**

For $x \neq 0$,

$$\widehat{U_{[c,d]}}(x) = \frac{1}{d-c}\int_c^d e^{ixy} d\lambda(y) = \frac{1}{d-c}\left[\frac{1}{ix}e^{ixy}\right]_c^d = \frac{1}{d-c}\frac{e^{ixd}-e^{ixc}}{ix}.$$

(3) **Fourier transform of convolution**

(a) **Gaussian distribution:** Letting $\mathbb{P}_{X_1}, \mathbb{P}_{X_2}$ be the laws of X_1, X_2, and $\varphi_{X_1}, \varphi_{X_2}$ the characteristic functions, we have

$$\widehat{\mathbb{P}_1}(x) = \varphi_{X_1}(x) = e^{i\langle m,x\rangle - \frac{1}{2}\langle Rx,x\rangle}, \qquad \widehat{\mathbb{P}_2}(x) = \varphi_{X_2}(x) = e^{i\langle \hat{m},x\rangle - \frac{1}{2}\langle \hat{R}x,x\rangle}.$$

Now since $\mathbb{P}_{X_1+X_2} = \mathbb{P}_{X_1} * \mathbb{P}_{X_2}$, we have

$$\varphi_{X_1+X_2}(x) = \widehat{\mathbb{P}_{X_1+X_2}}(x) = e^{i\langle m+\hat{m},x\rangle - \frac{1}{2}\langle (R+\hat{R})x,x\rangle}.$$

Therefore $X_1 + X_2$ is Gaussian with mean $m + \hat{m}$ and variance $R + \hat{R}$.

(b) **Uniform distribution:** Letting Y be uniform distributed on $[0,2]$, we see from Question 2 that for $x \neq 0$,

$$\varphi_{Y_1+Y_2}(x) = \left(\frac{e^{ix}-1}{ix}\right)^2, \qquad \varphi_Y(x) = \frac{e^{2ix}-1}{2ix}.$$

These functions are not equal in general, e.g.

$$\varphi_{Y_1+Y_2}\left(\frac{\pi}{2}\right) = -\frac{4}{\pi^2}(i-1)^2 \neq \frac{2i}{\pi} = \phi_Y\left(\frac{\pi}{2}\right).$$

(4) **linear image of measure**
Applying the change of variable formula and then the definition of the transpose,

$$\widehat{\mu_A}(x) = \int_{\mathbb{R}^d} e^{i\langle x,y\rangle} d\mu_A(y) = \int_{\mathbb{R}^d} e^{i\langle x,Ay\rangle} d\mu(y) = \int_{\mathbb{R}^d} e^{i\langle A^T x,y\rangle} d\mu(y) = \widehat{\mu}(A^T x).$$

(5) **square of characteristic function**
Let φ be the characteristic function of some random variable f, and let g be a random variable which is independent from f but with the same distribution (hence the same characteristic function φ). By Corollary 8.10.4,

$$\varphi_{f+g}(x) = \mathbb{E}(e^{i\langle x,f+g\rangle}) = \mathbb{E}(e^{i\langle x,f\rangle})\mathbb{E}(e^{i\langle x,g\rangle}) = \varphi(x)^2.$$

So φ^2 is the characteristic function of $f+g$.

(6) **Fourier transform of a compactly supported measure**
Since μ has compact support, we can fix $R>0$ large enough that $\mu(\mathbb{R}^d \setminus B(0,R)) = 0$. We now use the strategy from Theorem 11.2.6 (1) in the notes:

$$|\widehat{\mu}(x) - \widehat{\mu}(y)| \le \int_{B(0,R)} |e^{i\langle x,z\rangle} - e^{i\langle y,z\rangle}| d\mu(z) \le \int_{B(0,R)} |\langle x-y, z\rangle| d\mu(z) \le |x-y| \int_{B(0,R)} |z| d\mu(z) \le R|x-y|.$$

In the second inequality we used Sheet 5 Question 5c, and in the third inequality we used Cauchy–Schwarz.