Please hand in your answer to Question (1)* by email before the start of the exercise session, or on paper at the start of the exercise session.
You do not need to hand in the rest of your solutions (unless you are unable to attend the exercise session), but be prepared to present one or two of the ones you have completed (I will choose which) during the exercise session. If you are unable to attend the exercise session, please email me all the questions you have attempted by 10am on the day of the exercise session.

(1) * **Fourier transforms of discrete measures**

(a) **Finitely supported measure:** Show that for $a_1, \ldots, a_n \in \mathbb{R}^d$, $\theta_1, \ldots, \theta_n \in \mathbb{R}$, and $\mu = \sum_{k=1}^n \theta_k \delta_{a_k}$ considered as a signed measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, it holds that

$$\widehat{\mu}(x) = \sum_{k=1}^{n} \theta_k \big(\cos(\langle x, a_k \rangle) + i \sin(\langle x, a_k \rangle) \big) \qquad \text{for all } x \in \mathbb{R}^d,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^d$.

(b) **Poisson distribution:** Let $\lambda > 0$ and consider

$$\mathrm{Pois}_\lambda = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \delta_k$$

as a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Show that

$$\widehat{\mathrm{Pois}_\lambda}(x) = e^{\lambda(e^{ix}-1)}.$$

(2) **Fourier transform of uniform distribution**
Define

$$U_{[c,d]}(B) = \frac{\lambda(B \cap [c,d])}{d-c} \qquad \text{for } B \in \mathcal{B}(\mathbb{R}).$$

Show that for $x \neq 0$ it holds that

$$\widehat{U_{[c,d]}}(x) = \frac{1}{d-c} \frac{e^{ixd} - e^{ixc}}{ix}.$$

(3) **Fourier transform of convolution**
Use characteristic functions and properties of the convolution to show the following.

(a) **Gaussian distribution:** The sum of 2 independent random variables $X_1, X_2 \colon \Omega \to \mathbb{R}^d$ with Gaussian distribution (with expectation vectors m and $\hat{m}$ and covariance matrices R and $\hat{R}$, respectively) is again Gaussian distributed. Find the resulting expectation vector and covariance matrix.

(b) **Uniform distribution:** The sum of 2 independent random variables $Y_1, Y_2 \colon \Omega \to [0,1]$ which are uniformly distributed on $[0,1]$ does not have a uniform distribution on $[0,2]$.

(4) **linear image of measure**
With a $d \times d$ matrix A we describe the linear map $A \colon \mathbb{R}^d \to \mathbb{R}^d$. Assume $\mu \in \mathcal{M}_1^+(\mathbb{R}^d)$. Show that for the image measure μ_A given by

$$\mu_A(B) = \mu(\{x \in \mathbb{R}^d : Ax \in B\}) \qquad \text{for all } B \in \mathcal{B}(\mathbb{R}^d),$$

it holds that

$$\widehat{\mu_A}(x) = \widehat{\mu}(A^T x) \qquad \text{for all } x \in \mathbb{R}^d.$$

(5) **square of characteristic function**
If φ is a characteristic function, does it follow that φ^2 is also a characteristic function?
Hint: Compute the characteristic function of the sum of 2 independent random variables to get an idea.

(6) **Fourier transform of a compactly supported measure**
Let μ be a Borel probability measure on $\mathbb{R}^d$ whose support is compact (i.e. μ is supported inside a bounded subset of $\mathbb{R}^d$). Prove that $\widehat{\mu}$ is a Lipschitz continuous function, i.e. prove that there exists $C > 0$ such that for all $x, y \in \mathbb{R}^d$ it holds that $|\widehat{\mu}(x) - \widehat{\mu}(y)| \leq C|x - y|$.