

(1) * **Pólya's sufficient criteria**

One can check that f_1 and f_3 satisfy all four conditions in Pólya's theorem Corollary 16.2.2 (I will not write all the details here, but you should check them), so they are characteristic functions. We have $f_2(0) = 0 \neq 1$, and f_4 is not continuous, so by Theorem 11.2.6, f_2 and f_4 are not characteristic functions.

(2) **i.i.d. coin tosses**

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = \{H, T\}$, $\mathcal{F} = 2^{\{H, T\}}$, and $\mathbb{P}(\{H\}) = \mathbb{P}(\{T\}) = 1/2$ (this represents one toss of a fair coin). A sequence of independent tosses of a fair coin is represented by the infinite product probability space $(\times_{n=1}^{\infty} \Omega, \otimes_{n=1}^{\infty} \mathcal{F}, \otimes_{n=1}^{\infty} \mathbb{P})$. For $n \in \mathbb{N}$ define $X_n: \times_{n=1}^{\infty} \Omega \rightarrow \mathbb{R}$ by

$$X_n((\omega_1, \omega_2, \dots)) = \begin{cases} 1 & \text{if } \omega_n = H, \\ -1 & \text{if } \omega_n = T. \end{cases}$$

Then X_1, X_2, \dots are i.i.d. random variables with $\mathbb{E}(X_n) = 0$ and $\mathbb{E}(X_n^2) = 1 < \infty$. Therefore by the Classical Central Limit Theorem,

$$p_N = \mathbb{P}(X_1 + \dots + X_N \geq 2\sqrt{N}) = \mathbb{P}\left(\frac{X_1 + \dots + X_N}{\sqrt{N}} \geq 2\right) \xrightarrow{N \rightarrow \infty} p = \int_2^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \approx 0.023.$$

Since $\mathbb{E}(|X_n|^3) = 1 < \infty$, the Berry-Esseen theorem with $\alpha = 1$ gives that $|p_N - p| \leq c/\sqrt{N}$ for some $c > 0$ independent of N .

(3) **construction of an i.i.d. sequence with 3 values**

For all $k \in \mathbb{N}$ we have that $\{X_k = a_k\}$ is a union of 3^{k-1} intervals of length $1/3^k$, so is a Borel set. This shows that each X_k is $(\mathcal{B}([0, 1]), \mathcal{B}(\mathbb{R}))$ -measurable, and also shows that

$$\lambda(X_k = a_k) = 3^{k-1} \frac{1}{3^k} = \frac{1}{3}.$$

Moreover,

$$\lambda(X_1 = a_1, \dots, X_n = a_n) = \lambda\left(\left[\sum_{k=1}^n \frac{a_k}{3^k}, \sum_{k=1}^n \frac{a_k}{3^k} + \frac{1}{3^n}\right]\right) = \frac{1}{3^n}.$$

The result follows.

(4) **Gaussian - yes or no?**

Yes. Indeed, $|m_n - \mathbb{E}X| \leq \mathbb{E}|X_n - X| \rightarrow 0$, so $m := \mathbb{E}X \in \mathbb{R}$ and $m_n \rightarrow m$. Hence for all t , $e^{im_n t} \rightarrow e^{imt}$. Moreover, the L^1 convergence of $X_n \rightarrow X$ implies convergence in probability (Proposition 9.3.3), which in turn implies convergence in distribution (Corollary 12.1.9). Therefore for all $t \in \mathbb{R}$, by Theorem 11.7.1 (4),

$$e^{im_n t} e^{-\frac{1}{2}\sigma_n^2 t^2} = \varphi_{X_n}(t) \xrightarrow{n \rightarrow \infty} \phi_X(t).$$

Consequently $e^{-\frac{1}{2}\sigma_n^2 t^2}$ converges to some number in $[0, 1]$, so σ_n^2 converges to some number in $[0, \infty]$. But we cannot have $\sigma_n^2 \rightarrow \infty$, because then we would have $\varphi_{X_n}(t) \rightarrow 0$ for all $t \in \mathbb{R} \setminus \{0\}$, which would contradict the fact that φ_X is continuous with $\varphi_X(0) = 1$. So either $\sigma_n^2 \rightarrow 0$, in which case $X \sim \delta_m$ (which we defined to be Gaussian), or $\sigma_n^2 \rightarrow \sigma^2$ for some $\sigma \in (0, \infty)$, in which case $X \sim N_{m, \sigma^2}$.

(5) **Gaussian random variables which are pairwise independent, but not independent**

(a) First note that since X, Y are independent,

$$\mathbb{E}(\mathbb{1}_{XY \geq 0}) = \mathbb{P}(XY \geq 0) = \mathbb{P}(X \geq 0, Y \geq 0) + \mathbb{P}(X \leq 0, Y \leq 0) = \mathbb{P}(X \geq 0)\mathbb{P}(Y \geq 0) + \mathbb{P}(X \leq 0)\mathbb{P}(Y \leq 0) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}.$$

Therefore for all $x \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}(e^{iZx}) &= \mathbb{E}[e^{iZ_0x}(\mathbb{1}_{Z_0 \geq 0, XY \geq 0} + \mathbb{1}_{Z_0 < 0, XY < 0})] + \mathbb{E}[e^{-iZ_0x}(\mathbb{1}_{Z_0 \geq 0, XY < 0} + \mathbb{1}_{Z_0 < 0, XY \geq 0})] \\
&= \mathbb{E}(e^{iZ_0x} \mathbb{1}_{Z_0 \geq 0}) \mathbb{E}(\mathbb{1}_{XY \geq 0}) + \mathbb{E}(e^{iZ_0x} \mathbb{1}_{Z_0 < 0}) \mathbb{E}(\mathbb{1}_{XY < 0}) + \mathbb{E}(e^{-iZ_0x} \mathbb{1}_{Z_0 \geq 0}) \mathbb{E}(\mathbb{1}_{XY < 0}) + \mathbb{E}(e^{-iZ_0x} \mathbb{1}_{Z_0 < 0}) \mathbb{E}(\mathbb{1}_{XY \geq 0}) \\
&= \frac{1}{2} [\mathbb{E}(e^{iZ_0x} \mathbb{1}_{Z_0 \geq 0}) + \mathbb{E}(e^{iZ_0x} \mathbb{1}_{Z_0 < 0}) + \mathbb{E}(e^{-iZ_0x} \mathbb{1}_{Z_0 \geq 0}) + \mathbb{E}(e^{-iZ_0x} \mathbb{1}_{Z_0 < 0})] \\
&= \mathbb{E}(e^{iZ_0x}) = e^{-1/x^2},
\end{aligned}$$

where we used Theorem 11.7.1 (4) for the last equality. Therefore by the uniqueness theorem (Theorem 11.5.5), $Z \sim \mathcal{N}_{0,1}$.

(b) By assumption, X, Y are independent. By symmetry, it suffices to show that X, Z are independent. Let $A, B \in \mathcal{B}(\mathbb{R})$. Then

$$\begin{aligned}
\mathbb{P}(X \in A, Z \in B) &= \mathbb{P}(X \in A, Z \in B, X \geq 0) + \mathbb{P}(X \in A, Z \in B, X < 0) \\
&= \mathbb{P}(X \in A \cap [0, \infty), |Z_0| \text{sgn}(Y) \in B) + \mathbb{P}(X \in A \cap (-\infty, 0), -|Z_0| \text{sgn}(Y) \in B) \\
&= \mathbb{P}(X \in A \cap [0, \infty)) \mathbb{P}(|Z_0| \text{sgn}(Y) \in B) + \mathbb{P}(X \in A \cap (-\infty, 0)) \mathbb{P}(-|Z_0| \text{sgn}(Y) \in B) \\
&= \mathbb{P}(X \in A) \mathbb{P}(Z \in B).
\end{aligned}$$

To justify the last equality we use

$$\mathbb{P}(|Z_0| \text{sgn}(Y) \in B) = \mathbb{P}(|Z_0| \text{sgn}(XY) \in B, X \geq 0) + \mathbb{P}(-|Z_0| \text{sgn}(XY) \in B, X < 0) = \frac{1}{2} (\mathbb{P}(Z \in B) + \mathbb{P}(Z \in -B)) = \mathbb{P}(Z \in B),$$

and similarly $\mathbb{P}(-|Z_0| \text{sgn}(Y) \in B) = \mathbb{P}(Z \in B)$.

(c) We have $\mathbb{P}(Z \geq 0) = \mathbb{P}(XY \geq 0) = 1/2$, so

$$\mathbb{P}(X \geq 0, Y \geq 0, Z \geq 0) = \mathbb{P}(X \geq 0, Y \geq 0) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \neq \frac{1}{8} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \mathbb{P}(X \geq 0) \mathbb{P}(Y \geq 0) \mathbb{P}(Z \geq 0).$$