Please hand in your answer to Question (1)* by email before the start of the exercise session, or on paper at the start of the exercise session.
You do not need to hand in the rest of your solutions (unless you are unable to attend the exercise session), but be prepared to present one or two of the ones you have completed (I will choose which) during the exercise session. If you are unable to attend the exercise session, please email me all the questions you have attempted by 10am on the day of the exercise session.

(1) * **Pólya's sufficient criteria**
Argue which of the functions $f_k \colon \mathbb{R} \to \mathbb{R}$ is a characteristic function.
(a) $f_1(t) = e^{-|t|}$
(b) $f_2(t) = |t|$
(c) $f_3(t) = \frac{1}{1+|t|}$
(d) $f_4(t) = \mathbb{1}_{[-1,1]}(t)$
Hint: You can use Pólya's theorem (Corollary 16.2.2).

(2) **i.i.d. coin tosses**
Consider an infinite sequence of independent tosses of a fair coin. For each $N \in \mathbb{N}$, let p_N denote the probability that at least $N/2 + \sqrt{N}$ of the first N coin tosses are Heads. Explain why p_N converges to some number $p \in [0, 1]$ as $N \to \infty$, and write down an integral which describes p. Use a computer (e.g. Wolfram Alpha online) to compute the integral to 3 decimal places. What can one say about the rate of convergence of $p_N \to p$?
Hint: you can use the same setup as in Sheet 3 Question 2 (the solutions are on the course website), but instead of applying the law of large numbers use different theorems.

(3) **construction of an i.i.d. sequence with 3 values**
(The objective of this question is to construct an i.i.d. sequence $(X_n)_{n=1}^{\infty}$ of random variables on the probability space $([0, 1), \mathcal{B}([0, 1)), \lambda)$ with

$$\mathbb{P}(X_1 = 0) = \mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = 2) = \frac{1}{3}.)$$

For $t \in [0, 1)$ we use the representation

$$t = \sum_{n=1}^{\infty} \frac{t_n}{3^n},\ t_n \in \{0, 1, 2\},$$

where for the sake of its uniqueness we agree that $(t_n)_{n=1}^{\infty}$ does not have an infinite string of 2s. Put

$$X_n(t) \coloneqq t_n.$$

Show that for all $a_1, \ldots, a_n \in \{0, 1, 2\}$ we have

$$\begin{aligned} \lambda(\{t \in [0, 1) : X_1(t) = a_1, \ldots, X_n(t) = a_n\}) &= \frac{1}{3^n} \\ &= \lambda(\{t \in [0, 1) : X_1(t) = a_1\}) \cdots \lambda(\{t \in [0, 1) : X_n(t) = a_n\}). \end{aligned}$$

(4) **Gaussian - yes or no?**
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Assume that $X, X_1, X_2, \ldots : \Omega \to \mathbb{R}$ are random variables such that X_k converges to X in L_1, i.e. $\mathbb{E}|X_k - X| \to 0$ as $k \to \infty$. If each random variable X_k is Gaussian with expectation m_k and variance σ_k^2, does it then follow that X is Gaussian?

(5) **Gaussian random variables which are pairwise independent, but not independent**
Let X, Y and Z_0 denote i.i.d. standard normal distributed random variables. Define

$$Z := |Z_0| \operatorname{sgn}(XY),$$

where

$$\operatorname{sgn}(x) = \begin{cases} 1 \text{ if } x \geq 0 \\ -1 \text{ if } x < 0. \end{cases}$$

Show that
(a) Z is standard normal distributed,
(b) X, Y, Z are pairwise independent,
(c) X, Y, Z are not independent.