Intermediate dimensions

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[1] Includes joint work with:
- Haipeng Chen: https://arxiv.org/abs/2212.06961, J. Fractal Geom.
- István Kolossváry: https://arxiv.org/abs/2111.05625, Preprint
- Alex Rutar: https://arxiv.org/abs/2111.14678, Ann. Fenn. Math.

Hausdorff dimension

- Different notions of dimension attempt to quantify the 'thickness' of fractal sets at small scales.
- Hausdorff dimension can be defined without using Hausdorff measure:

$$\dim_{\mathrm{H}} F = \inf\{\, s \geq 0 : \text{for all } \varepsilon > 0 \text{ there exists a finite or countable cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } \sum_i |U_i|^s \leq \varepsilon \,\}$$

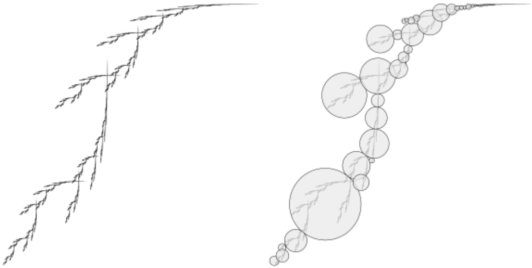


Figure: A cover using balls of different sizes. Picture by Jonathan Fraser.

Box dimension

- Upper box dimension is defined by

$$\overline{\dim}_{\mathrm{B}} F := \limsup_{\delta \to 0^+} \frac{\log N_\delta(F)}{-\log \delta},$$

where $N_\delta(F)$ is the smallest number of balls of radius δ needed to cover F.

- Alternative definition:

$$\overline{\dim}_{\mathrm{B}} F = \inf\{\, s \geq 0 : \text{for all } \varepsilon > 0 \text{ there exists } \delta_0 \in (0,1] \text{ such that for all } \delta \in (0, \delta_0) \text{ there exists a cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } |U_i| = \delta \text{ for all } i, \text{ and } \sum_i |U_i|^s \leq \varepsilon \,\}.$$

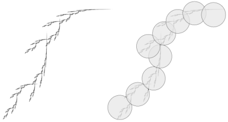


Figure: A cover using balls of the same size. Picture by Jonathan Fraser.

- Always $\dim_{\mathrm{H}} F \leq \overline{\dim}_{\mathrm{B}} F$.

Intermediate dimensions

Falconer, Fraser and Kempton ('20) defined the upper θ-intermediate dimension for $\theta \in (0,1)$, satisfying

$$\dim_{\mathrm{H}} F \le \overline{\dim}_{\theta} F \le \overline{\dim}_{\mathrm{B}} F.$$

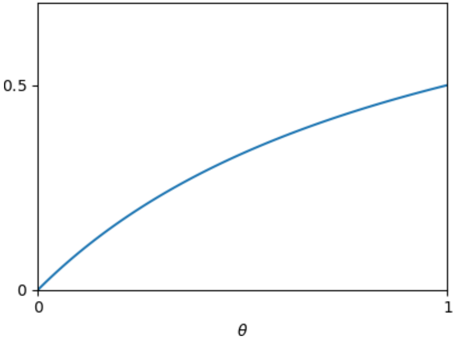
$$\overline{\dim}_{\theta} F := \inf\{\, s \ge 0 : \text{for all } \varepsilon > 0 \text{ there exists } \delta_0 \in (0,1] \text{ such that for all } \delta \in (0,\delta_0) \text{ there exists a cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } \delta^{1/\theta} \le |U_i| \le \delta \text{ for all } i, \text{ and } \sum_i |U_i|^s \le \varepsilon \,\}.$$

Example of dimension interpolation (see also Assouad spectrum, Fraser–Yu, '18).

Polynomial sequences

- For $p \in (0, \infty)$ define $F_p := \{ n^{-p} : n \in \mathbb{N} \}$.

- These sets satisfy $\dim_{\mathrm{H}} F_p = 0$, $\dim_{\mathrm{B}} F_p = \frac{1}{p+1}$.
- Falconer–Fraser–Kempton ('20) showed that $\dim_\theta F_p = \frac{\theta}{p+\theta}$.



- Proof: cover with intervals of size δ until a certain point, then cover each dot with size $\delta^{1/\theta}$.

Popcorn function

The popcorn function $f\colon (0,1) \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ where } 1 \leq p < q, \gcd(p,q) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

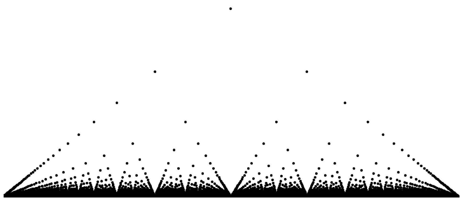


Figure: The graph $P \subset \mathbb{R}^2$ of the popcorn function

Theorem (Chen–Fraser–Yu, '22)

$\dim_{\mathrm{B}} P = 4/3$

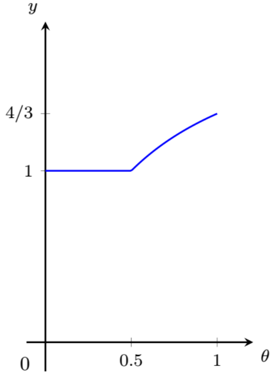
Proved using techniques from Diophantine approximation and probability theory.

Popcorn function

Theorem (B.–Chen, '22+)

The intermediate dimensions satisfy

$$\dim_\theta P = \begin{cases} 1 & 0 \leq \theta \leq 1/2 \\ \frac{4\theta}{2\theta+1} & 1/2 < \theta \leq 1. \end{cases}$$



Iterated function systems (IFSs)

- Let $X \subset \mathbb{R}^d$ be compact and let $\{S_i \colon X \to X\}_{i \in I}$ be a finite set of contractions (an IFS). By Hutchinson ('81) there is a unique non-empty compact attractor F satisfying $F = \bigcup_{i \in I} S_i(F)$.
- In this talk the open set condition is always satisfied: there exists a non-empty bounded open set U with $\bigcup_{i \in I} S_i(U) \subseteq U$ with the union disjoint.
- If each S_i is assumed to be a similarity map with contraction ratio c_i then the Hausdorff and box dimensions of F coincide with the unique $h \geq 0$ satisfying Hutchinson's formula

$$\sum_{i \in I} c_i^h = 1.$$

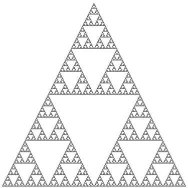
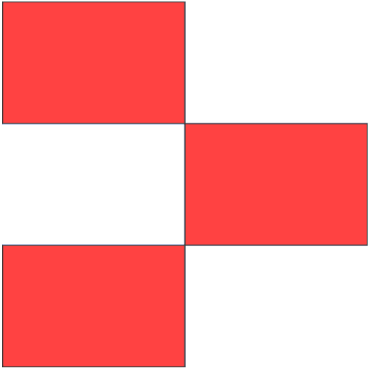


Figure: The Sierpinski gasket has dimension $\log 3 / \log 2$.

Bedford–McMullen carpets

If the contractions are affine then the dimensions can differ.
Divide a square into an $m \times n$ grid, $2 \leq m < n$.



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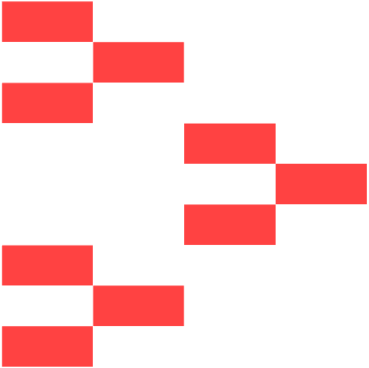

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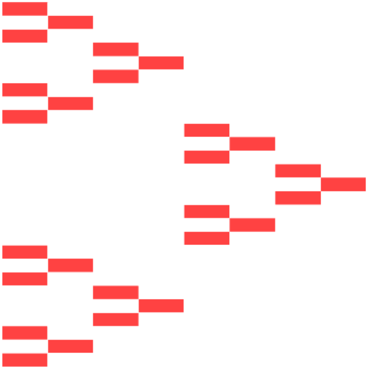

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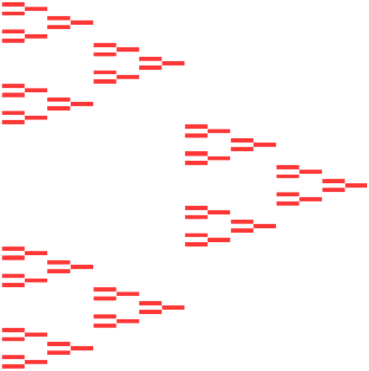

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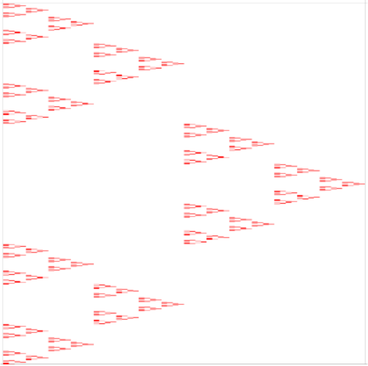


Hausdorff and box dimensions

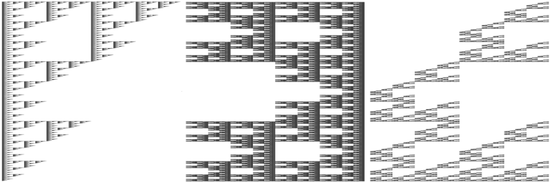


Figure: Three different Bedford–McMullen carpets. Picture by Jonathan Fraser.

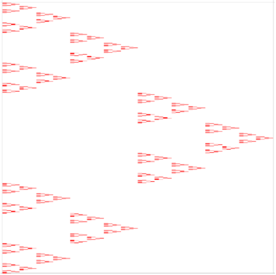
Theorem (Bedford '84, McMullen '84)

$$\dim_{\mathrm{H}} \Lambda = \frac{1}{\log m} \log \left(\sum_{\hat{\imath}=1}^{M} N_{\hat{\imath}}^{\frac{\log m}{\log n}} \right); \qquad \dim_{\mathrm{B}} \Lambda = \frac{\log M}{\log m} + \frac{\log(N/M)}{\log n}.$$

Here, $M := \#$ non-empty columns, $N_i := \#$ maps in ith non-empty column, $N := N_1 + \cdots + N_M$.

Hausdorff and box dimensions differ iff the carpets have non-uniform vertical fibres (which we always assume).

Graph of the intermediate dimensions



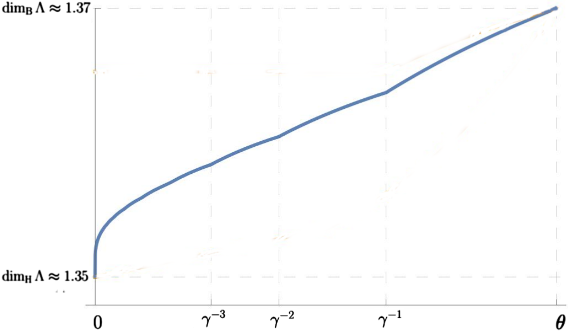


Figure: Here, $\gamma := \log n / \log m$

Formula for the intermediate dimensions (B.–Kolossváry, '21+)

- Define the Legendre transform

$$I(t) := \sup_{\lambda\in\mathbb{R}} \left\{ \lambda t - \log\left(\frac{1}{M} \sum_{\hat{\jmath}=1}^{M} N_{\hat{\jmath}}^{\lambda} \right) \right\}.$$

- For $s \in \mathbb{R}$, define the function $T_s \colon \mathbb{R} \to \mathbb{R}$ by

$$T_s(t) := \left(s - \frac{\log M}{\log m} \right) \log n + \gamma I(t).$$

- For $\ell \in \mathbb{N}$, write $T_s^\ell := \underbrace{T_s \circ \cdots \circ T_s}_{\ell \text{ times}}$, and T_s^0 is the identity. Define

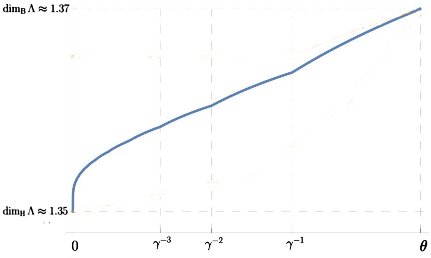
$$t_\ell(s) := T_s^{\ell-1}\left(\left(s - \frac{\log M}{\log m} \right) \log n \right).$$

- For fixed $\theta \in (0,1)$ let $L = L(\theta) \in \mathbb{N}$ be such that $\gamma^{-L} < \theta \leq \gamma^{-(L-1)}$. Then $\dim_\theta \Lambda$ is the unique solution $s = s(\theta) \in (\dim_{\mathrm{H}} \Lambda, \dim_{\mathrm{B}} \Lambda)$ to the equation

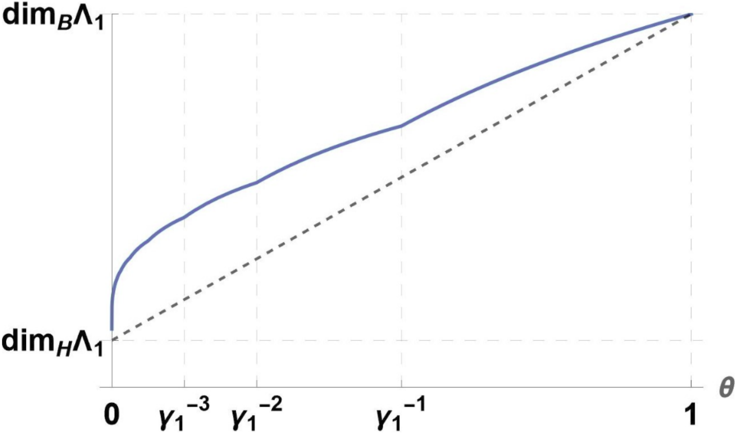
$$\gamma^L \theta \log N - (\gamma^L\theta - 1) t_L(s) + \gamma(1 - \gamma^{L-1}\theta)(\log M - I(t_L(s))) - s \log n = 0.$$

Intermediate dimensions of Bedford–McMullen carpets

- Phase transitions at negative integer powers of $\log n / \log m$.
- Real analytic and strictly concave between phase transitions
- Strictly increasing
- Right derivative tends to ∞ as $\theta \to 0$
- Continuous for $\theta \in [0, 1]$ (proved by Falconer–Fraser–Kempton ('20), used by Burrell–Falconer–Fraser ('21) to prove results on the box dimension of orthogonal projections of carpets)



Different possible shapes of the graph



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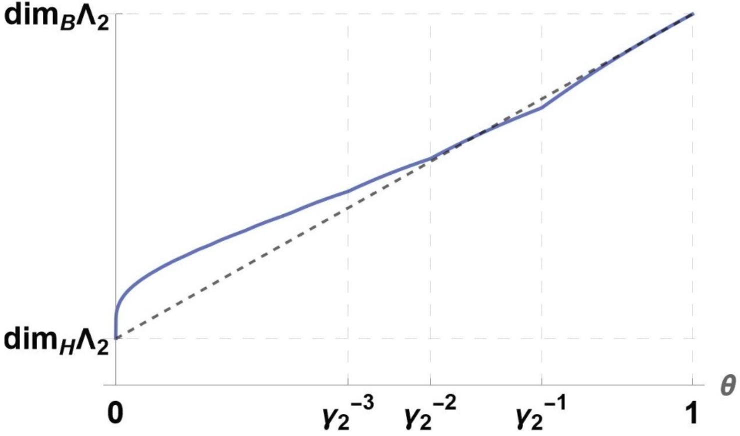

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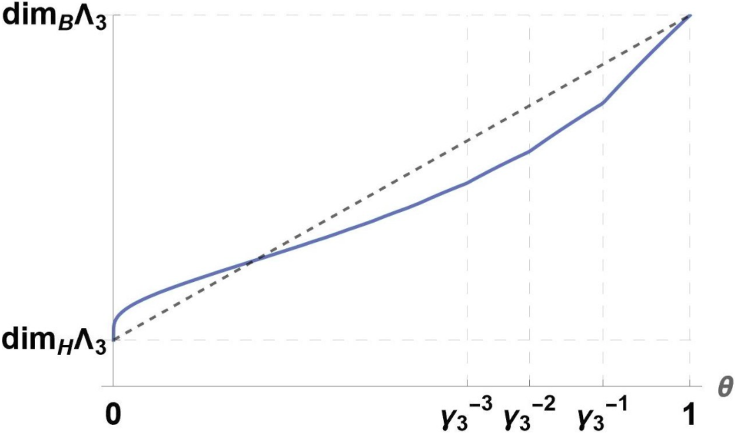

Ingredients for the proof

- Upper bound: construct an intricate cover using scales $\delta, \delta^{\gamma}, \delta^{\gamma^2}, \ldots, \delta^{\gamma^{L-1}}$ and $\delta^{1/\theta}, \delta^{1/(\gamma\theta)}, \ldots, \delta^{1/(\gamma^{L-1}\theta)}$ (we need to use more than two scales when θ is small).
- The proof simplifies substantially when $\theta \geq 1/\gamma$ (use just largest and smallest scales) or $\theta = \gamma^{-k}$ (use scales $\delta, \delta^{\gamma}, \ldots, \delta^{\gamma^k}$).
- A cylinder of length δ has height $\approx \delta^{\gamma}$. We break the carpet into approximate square of size δ by stacking such cylinders. We cover depending on how parts of the symbolic representation of the approximate square relate to each other, using the method of types.
- Lower bound uses a variant of a mass distribution principle proved by Falconer, Fraser and Kempton ('20).

Multifractal analysis

- Let ν be the uniform Bernoulli measure supported on a Bedford–McMullen carpet, satisfying

$$\nu(A) = \sum_{i=1}^{N} \frac{1}{N} \nu(S_i^{-1} A) \text{ for all Borel sets } A \subset \mathbb{R}^2.$$

  where N is the total number of contractions.

- Kenyon and Peres ('96) showed that ν is exact dimensional: the local dimension

$$\dim_{\mathrm{loc}}(\nu, x) = \lim_{r \to 0} \frac{\log \nu(B(x,r))}{\log r}$$

  exists and is constant at ν-almost every $x \in \Lambda$.

- Jordan and Rams ('11) computed the multifractal spectrum of ν,

$$f_\nu(\alpha) := \dim_{\mathrm{H}} \{ x \in \operatorname{supp} \nu : \dim_{\mathrm{loc}}(\nu, x) = \alpha \},$$

  building on work of King ('95).

Multifractal analysis and bi-Lipschitz equivalence

Theorem (B.–Kolossváry, '21+)

If Λ, Λ' are Bedford–McMullen carpets with non-uniform vertical fibres, then the intermediate dimensions are equal for all θ if and only if the corresponding uniform Bernoulli measures have the same multifractal spectra.

If $f\colon \Lambda \to \Lambda'$ is bi-Lipschitz then $\dim_\theta \Lambda = \dim_\theta \Lambda'$ for all θ.

Corollary

If carpets Λ and Λ' with non-uniform vertical fibres are bi-Lipschitz equivalent then their uniform Bernoulli measures have the same multifractal spectra.

This improves a result of Rao, Yang and Zhang ('21+).

Attainable forms of intermediate dimensions

The upper Dini derivative of a function $h\colon \mathbb{R} \to \mathbb{R}$ at x is given by

$$D^+ h(x) = \limsup_{\varepsilon \to 0^+} \frac{h(x+\varepsilon) - h(x)}{\varepsilon}.$$

Theorem (B.–Rutar, '22)

Let $h\colon [0,1] \to [0,d]$ be any function. Then there exists a non-empty bounded set $F \subset \mathbb{R}^d$ with $\dim_\theta F = h(\theta)$ if and only if h is non-decreasing, is continuous on $(0,1]$, and satisfies

$$D^+ h(\theta) \leq \frac{h(\theta)(d - h(\theta))}{d\theta} \qquad \text{for all } \theta \in (0,1).$$

Proof idea: Necessity: 'Break up' the largest sets in the cover and 'fatten' the smallest ones, to get a new cover corresponding to larger θ.
Sufficiency: construct an appropriate homogeneous Moran set M and prove that $\overline{\dim}_\theta M = \limsup_{\delta \to 0}(\inf_{\phi \in [\delta^{1/\theta}, \delta]} \frac{\log N_\phi(M)}{-\log \phi})$

Attainable forms of intermediate dimensions

Consequence: if f is any non-decreasing Lipschitz function on $[0, 1]$, there exist $a > 0, b \in \mathbb{R}$ and $F \subset \mathbb{R}$ such that $\dim_\theta F = af(\theta) + b$. In particular, the following are possible:

- Strictly increasing then constant then strictly increasing.
- Strictly convex, strictly concave or linear.
- Non-differentiable at each point in a dense subset of $[0, 1]$.

Recovering the interpolation

If $F_{\log} := \{0\} \cup \{\, 1/(\log n) : n \in \mathbb{N} \,\}$, then $\dim_\theta F_{\log} = 1$ for all $\theta \in (0,1]$, so there is not full interpolation.

Theorem (B., '20)

If $F \subset \mathbb{R}^d$ is non-empty and compact then for all $s \in [\dim_{\mathrm{H}} F, \overline{\dim}_{\mathrm{B}} F]$ there exists a function $\Phi_s \colon (0,1) \to (0,1)$ that is monotonic and satisfies $\Phi_s(\delta) \leq \delta$ for all δ, such that if we define

$$\overline{\dim}^{\Phi_s} F = \inf\{\, s \geq 0 : \text{for all } \varepsilon > 0 \text{ there exists } \delta_0 \in (0,1] \text{ such that for all } \delta \in (0,\delta_0) \text{ there exists a cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } \Phi_s(\delta) \leq |U_i| \leq \delta \text{ for all } i, \text{ and } \sum_i |U_i|^s \leq \varepsilon \,\}$$

then $\overline{\dim}^{\Phi_s} F = s$.

Proof idea: define

$$\Phi_s(\delta) := \sup\{\, x \in [0,\delta] : \text{there exists a finite cover } \{U_i\} \text{ of } F \text{ such that } x \leq |U_i| \leq \delta \text{ for all } i \text{ and } \sum_i |U_i|^s \leq 1 \,\}.$$

Thank you for listening!

谢谢大家

Questions welcome