Dimensions of infinitely generated self-conformal sets

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[1]Based on work in:
- ‘Intermediate dimensions of infinitely generated attractors’ (with Jonathan M. Fraser), **Trans. Amer. Math. Soc.** (to appear), `https://arxiv.org/abs/2104.15133`, 2021.
- ‘Assouad type dimensions of infinitely generated self-conformal sets’ (with Jonathan M. Fraser), Preprint, `https://arxiv.org/abs/2207.11611`, 2022.

Fractals and dimension

- Fractals typically have fine structure at arbitrarily small scales, and some sort of self-similarity.
- Different notions of dimension attempt to quantify the ‘thickness’ of sets at small scales.

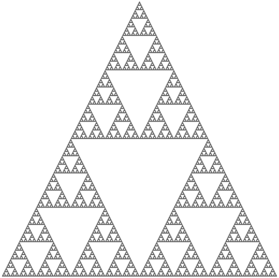


Figure: The Sierpinski gasket is a fractal which has dimension ≈ 1.58

Hausdorff content

- Throughout, $F \subset \mathbb{R}^n$ will be non-empty and bounded.
- For $s \geq 0$ and $\delta > 0$, define the Hausdorff content

$$H^s_\delta(F) = \inf\left\{ \sum_{i=1}^{\infty} |U_i|^s \,\middle|\, F \subseteq \bigcup_i U_i, \mathrm{diam}(U_i) \leq \delta \right\}.$$

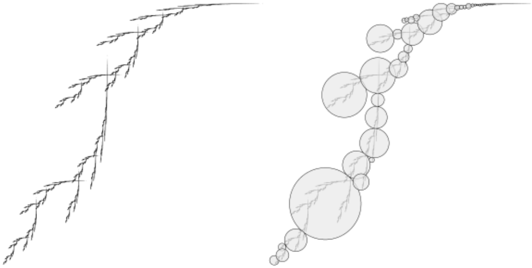


Figure: A cover using balls of different sizes. Picture credit: Jonathan Fraser.

Hausdorff dimension

- As δ decreases, the infimum increases, so converges to a limit

$$H^s_\delta(F) \to H^s(F) \in [0,\infty] \text{ as } \delta \to 0^+,$$

  called the *s*-dimensional Hausdorff measure of F.
- For each F there is a unique $s \geq 0$, called the Hausdorff dimension of F, such that if $0 \leq t < s$ then $H^t(F) = \infty$ and if $t > s$ then $H^t(F) = 0$.

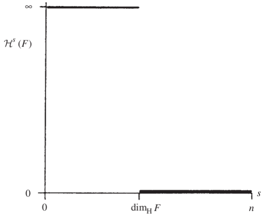


Figure: Graph of the s-dimensional Hausdorff measure of a set against s. Picture credit: Kenneth Falconer.

- Intuitively, disc has Hausdorff dimension 2 because it has positive and finite area.

Box dimension

- The (upper) box dimension is defined by

$$\overline{\dim}_{\mathrm{B}} F := \limsup_{\delta \to 0^+} \frac{\log N_\delta(F)}{-\log \delta},$$

where $N_\delta(F)$ is the smallest number of balls of radius δ needed to cover F.

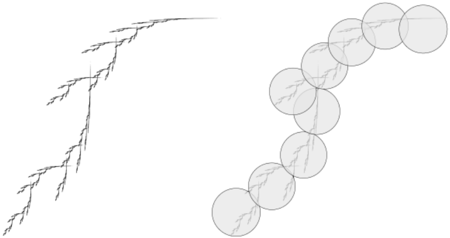


Figure: A cover using balls of the same size. Picture credit: Jonathan Fraser.

- Intuitively, a disc has box dimension 2 because the number of discs of size δ needed to cover it scales approximately like δ^{-2} as $\delta \to 0^+$.

Assouad dimension

- Assouad dimension is the largest reasonable notion of dimension. It captures the scaling behaviour of the 'thickest' parts of the set.

$$\dim_A F := \inf \{ \alpha : \text{ there exists } C > 0 \text{ such that for all } x \in F \text{ and } 0 < r < R, \text{ we have } N_r(B(x,R) \cap F) \leq C(R/r)^\alpha \}.$$

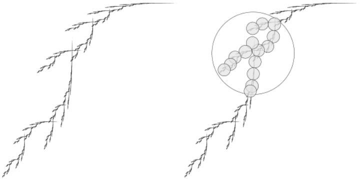


Figure: Covering a ball for the Assouad dimension. Picture credit: Jonathan Fraser.

- It has applications to embeddability problems.
- In general,

$$\dim_{\mathrm{H}} F \leq \overline{\dim}_{\mathrm{B}} F \leq \dim_{\mathrm{A}} F.$$

Iterated function systems (IFSs)

- An IFS is a finite set of contractions $\{S_i\colon X \to X\}_{i \in I}$ where $X \subset \mathbb{R}^n$ is compact.
- Hutchinson (1981) showed there is a unique non-empty attractor/limit set satisfying

$$F = \bigcup_{i \in I} S_i(F).$$

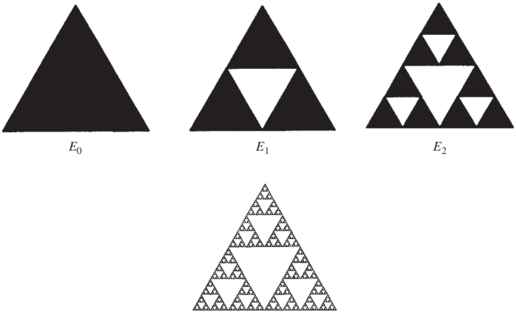


Figure: The construction of the Sierpinski gasket. Picture credit: Kenneth Falconer.

Self-similar sets

- If each of the contractions S_i is a similarity (so there exists $c_i \in (0,1)$ such that $||S_i(x) - S_i(y)|| = c_i||x - y||$ for all $x, y \in X$) then F is a self-similar set.
- We assume the open set condition (OSC), which asserts that $\text{Int}(X) \neq \varnothing$ and

$$\bigcup_{i \in I} S_i(\text{Int}(X)) \subseteq \text{Int}(X)$$

  with the union disjoint.
- Then the Hausdorff, box and Assouad dimensions all equal the unique $h \geq 0$ such that

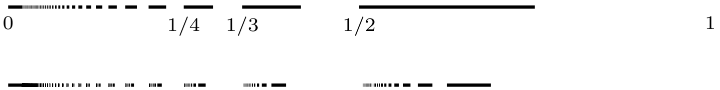
$$\sum_{i \in I} c_i^h = 1,$$

  and the set is very homogeneous. For example, the dimension of the Sierpiński gasket is $\log 3/\log 2$.

Conformal maps

- Conformal maps locally preserve angles.
- If $V \subseteq \mathbb{R}^n$ is open then $f\colon V \to \mathbb{R}^n$ is conformal if for all $x \in V$ the differential $Df|_x$ exists, is non-zero, is Hölder continuous in x, and is a similarity map: $||Df|_x(y)|| = ||Df|_x|| \cdot ||y||$ for all $y \in \mathbb{R}^n$.
- In one dimension, they are simply functions with non-vanishing Hölder continuous derivative.
  In two dimensions, they are holomorphic functions with non-vanishing derivative on their domain.
  In dimension three and higher, by Liouville (1850) they are Möbius transformations.

Infinite conformal iterated function systems (Mauldin–Urbański, '96)



First and second level cylinders for an infinitely generated self-similar set

An (infinite) conformal iterated function system (CIFS) is a countable number of maps $\{S_i \colon X \to X\}_{i \in I}$ that satisfies the following properties:

- Conformality: There exists an open, bounded, connected subset $V \subset \mathbb{R}^n$ such that $X \subset V$ and such that each S_i extends to a conformal map from V to an open subset of V. Moreover, there exists $\rho \in (0,1)$ such that $||S_i'||_\infty < \rho$ for all $i \in I$.
- Open set condition
- Cone condition: $\inf_{x \in X} \inf_{r \in (0,1)} \mathcal{L}^d(B(x,r) \cap \mathrm{Int}_{\mathbb{R}^d} X)/r^d > 0$.
- Bounded distortion property: There exists $K > 0$ such that for all $x, y \in X$ and any finite word $w = (i_1, \ldots, i_k)$ we have $||S_w'|_y|| \leq K ||S_w'|_x||$, where $S_w := S_{i_i} \circ \cdots \circ S_{i_k}$.

Limit set

- The limit set F of a CIFS can be defined as the largest set (by inclusion) which satisfies
$$F = \bigcup_{i \in I} S_i(F).$$
- It is non-empty but is not generally closed.
- It is well known that if I is finite then the Hausdorff, box and Assouad dimensions of F coincide.
If I is infinite then they can all differ, because the box and Assouad dimensions can be influenced by the countable set of fixed points.

Pressure function

- For $w \in I^k$ define

$$R_w := \sup_{x,y \in X, x \neq y} \frac{||S_w(x) - S_w(y)||}{||x - y||},$$

  the smallest possible Lipschitz constant for S_w. For $v, w \in I^*$ we have $R_{vw} \leq R_v R_w$, so the sequence $\left(\log \sum_{w \in I_k} R_w^t\right)_{k \in \mathbb{N}}$ is subadditive.

- Therefore we can define the topological pressure function $\overline{P}\colon (0, \infty) \to [-\infty, \infty]$ by

$$\overline{P}(t) := \lim_{k \to \infty} \frac{1}{k} \log \sum_{w \in I_k} R_w^t.$$

Hausdorff and box dimensions

Henceforth, F will be the limit set of a CIFS.

Theorem (Mauldin–Urbański, '96)

$\dim_{\mathrm{H}} F = \inf\{\, t > 0 : \overline{P}(t) < 0 \,\}$

- In particular, if each S_i is a similarity with contraction ratio c_i then $\dim_{\mathrm{H}} F = \inf\{\, t \geq 0 : \sum_{i \in I} c_i^t \leq 1 \,\}$.
- There may not exist $t \geq 0$ such that $\overline{P}(t) = 0$.

Here and later, P denotes the set of fixed points of the contractions:

Theorem (Mauldin–Urbański, '99)

$\overline{\dim}_{\mathrm{B}} F = \max\{\dim_{\mathrm{H}} F, \overline{\dim}_{\mathrm{B}} P\}$

Assoud dimension

Theorem (B.–Fraser, '22+)

Assuming $S_i(V) \cap S_j(V) = \varnothing$ for all distinct $i, j \in I$,

$$\dim_{\mathrm{A}} F = \max\{\dim_{\mathrm{H}} F, \dim_{\mathrm{A}} P\}.$$

Question: does one really need to assume the additional separation condition?

Dimension interpolation

- Take two notions of dimension dim and Dim for which $\dim F \leq \operatorname{Dim} F$ for all 'reasonable' sets F. Try to find a geometrically natural family of dimensions that always lies between them.
- Intermediate dimensions (Falconer–Fraser–Kempton, '20): for $\theta \in (0,1)$,

$$\dim_{\mathrm{H}} F \leq \overline{\dim}_{\theta} F \leq \overline{\dim}_{\mathrm{B}} F.$$

- Assouad spectrum (Fraser–Yu, '18): for $\theta \in (0,1)$,

$$\overline{\dim}_{\mathrm{B}} F \leq \dim_{\mathrm{A}}^{\theta} F \leq \dim_{\mathrm{A}} F.$$

Intermediate dimensions

- Hausdorff dimension:

$$\dim_{\mathrm{H}} F = \inf\{\, s \geq 0 : \text{for all } \varepsilon > 0 \text{ there exists a finite or countable cover } \{U_1, U_2, \dots\} \text{ of } F \text{ such that } \sum_i |U_i|^s \leq \varepsilon \,\}$$

- Box dimension:

$$\overline{\dim}_{\mathrm{B}} F = \inf\{\, s \geq 0 : \text{for all } \varepsilon > 0 \text{ there exists } \delta_0 \in (0,1] \text{ such that for all } \delta \in (0, \delta_0) \text{ there exists a cover } \{U_1, U_2, \dots\} \text{ of } F \text{ such that } |U_i| = \delta \text{ for all } i, \text{ and } \sum_i |U_i|^s \leq \varepsilon \,\}.$$

- Upper θ-intermediate dimension for $\theta \in (0,1)$:

$$\overline{\dim}_{\theta} F = \inf\{\, s \geq 0 : \text{for all } \varepsilon > 0 \text{ there exists } \delta_0 \in (0,1] \text{ such that for all } \delta \in (0, \delta_0) \text{ there exists a cover } \{U_1, U_2, \dots\} \text{ of } F \text{ such that } \delta^{1/\theta} \leq |U_i| \leq \delta \text{ for all } i, \text{ and } \sum_i |U_i|^s \leq \varepsilon \,\}.$$

Intermediate dimensions

Theorem (B.–Fraser, '21+)

$\overline{\dim}_\theta F = \max\{\dim_{\mathrm{H}} F, \overline{\dim}_\theta P\}$.

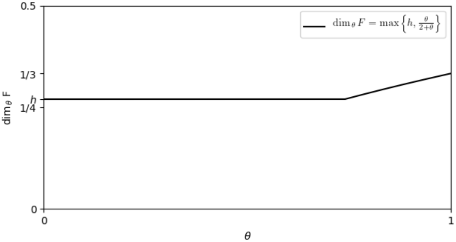


Figure: Intermediate dimensions when $P = \{ k^{-2} : k \in \mathbb{N} \}$

- Lower bounds are trivial. Upper bound for box and intermediate dimensions uses an induction argument.
- Using work of Burrell ('21+) we use the continuity of the intermediate dimensions to prove applications to box dimensions of orthogonal projections and fractional Brownian images.

Assoud spectrum

- Gives information about the thickest parts of the set with restriction on the relative scales according to θ.
- Assouad spectrum:

$$\dim_{\mathrm{A}}^{\theta} F := \inf \Big\{ \alpha : \text{ there exists } C > 0 \text{ such that for all } x \in F \text{ and } 0 < R < 1, r = R^{1/\theta}, \text{ we have } N_r(B(x,R) \cap F) \leq C(R/r)^{\alpha} \Big\}.$$

- Upper Assouad spectrum:

$$\overline{\dim}_{\mathrm{A}}^{\theta} F := \inf \Big\{ \alpha : \text{ there exists } C > 0 \text{ such that for all } x \in F \text{ and } 0 < R < 1, r \leq R^{1/\theta}, \text{ we have } N_r(B(x,R) \cap F) \leq C(R/r)^{\alpha} \Big\}.$$

Assouad spectrum

In general the Assouad spectrum of sets is not monotonic, but...

Lemma

If F is the limit set of a CIFS then the function $\theta \mapsto \dim_{\mathrm{A}} F$ is increasing in θ. In particular, $\dim_{\mathrm{A}}^{\theta} F = \overline{\dim}_{\mathrm{A}}^{\theta} F$.

Theorem (B.–Fraser, '22+)

$$\max\{\dim_{\mathrm{H}} F, \overline{\dim}_{\mathrm{A}}^{\theta} P\} \leq \dim_{\mathrm{A}}^{\theta} F \leq \max_{\phi \in [\theta,1]} f(\theta,\phi),$$

where for $\theta \in (0,1)$ and $\phi \in (0,1]$,

$$f(\theta,\phi) := \frac{(\phi^{-1}-1)\overline{\dim}_{\mathrm{A}}^{\phi} P + (\theta^{-1}-\phi^{-1})\overline{\dim}_{\mathrm{B}} F}{\theta^{-1}-1}.$$

In particular, $f(\theta,\theta) = \overline{\dim}_{\mathrm{A}}^{\theta} P$ and $f(\theta,1) = \overline{\dim}_{\mathrm{B}} F$.

Assoaud spectrum - example

If $P = \{k^{-p} : k \in \mathbb{N}\}$ then Fraser and Yu ('18) proved that

$$\dim_{\mathrm{A}}^{\theta} P = \min\left\{\frac{1}{(1+p)(1-\theta)}, 1\right\}$$

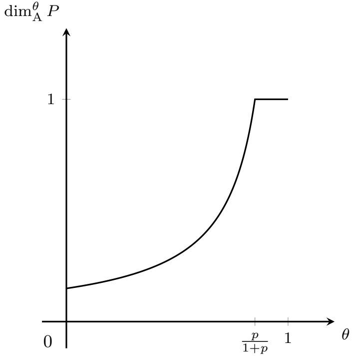


Figure: Case $p = 5.7$

Assoaud spectrum - example

If $p = 5.7$ and the first finitely many contraction ratios are chosen so that $h = 0.45$ then the upper bound is

$$\dim_{\mathrm{A}}^{\theta} F \leq f\left(\theta, \frac{p}{1+p}\right) = \begin{cases} h + \frac{\theta}{p(1-\theta)}(1-h), & \text{for } 0 \leq \theta < \frac{p}{1+p} \\ 1, & \text{for } \frac{p}{1+p} \leq \theta \leq 1 \end{cases}$$

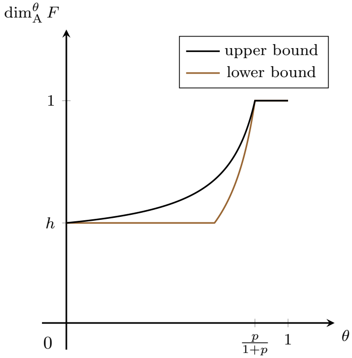


Figure: Bounds when $p = 5.7$

Assouad spectrum - example

Choosing the contraction ratios $c_k = k^{-t}$ for fixed $t \in [p+1, p+h^{-1}]$ and all large k shows that the bounds are sharp:

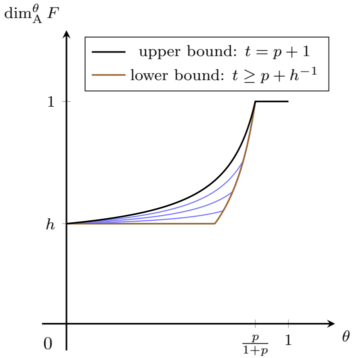
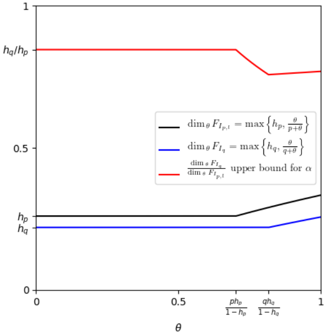


Figure: Graph of the Assouad spectrum for different values of t

These are the first dynamically generated fractals with Assouad spectrum having two phase transitions (elliptical polynomial spirals also do – see Burrell–Falconer–Fraser, '21+).

Lipschitz and Hölder maps

- If $g \colon X \to Y$ is bi-Lipschitz and dim is any of the dimensions mentioned today, then $\dim X = \dim Y$.
- The only one of these dimensions that can detect that different t give sets which are not bi-Lipschitz equivalent is the Assouad spectrum.
- If $g \colon X \to Y$ is α-Hölder then $\overline{\dim}_\theta g(X) \leq \alpha^{-1} \overline{\dim}_\theta X$, so the intermediate dimensions give upper bounds for α:



Continued fractions

- For $I \subseteq \mathbb{N}$ define

$$F_I := \left\{ z \in (0,1)\backslash\mathbb{Q} : z = \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{\ddots}}}, b_n \in I \text{ for all } n \in \mathbb{N} \right\}.$$

- Then $\{\, S_b(x) \coloneqq 1/(b+x) : b \in I \,\}$ is a CIFS (if $1 \notin I$) with limit set F_I.
- If the symmetric difference of I and $\{\, \lfloor n^p \rfloor : n \in \mathbb{N} \,\}$ is finite then the Assouad spectrum of F_I has the same form as the previous example with $t = 2p$.

Thank you for listening!

Questions welcome