SOLVABILITY OF PARTIAL DIFFERENTIAL EQUATIONS ON FRACTAL DOMAINS

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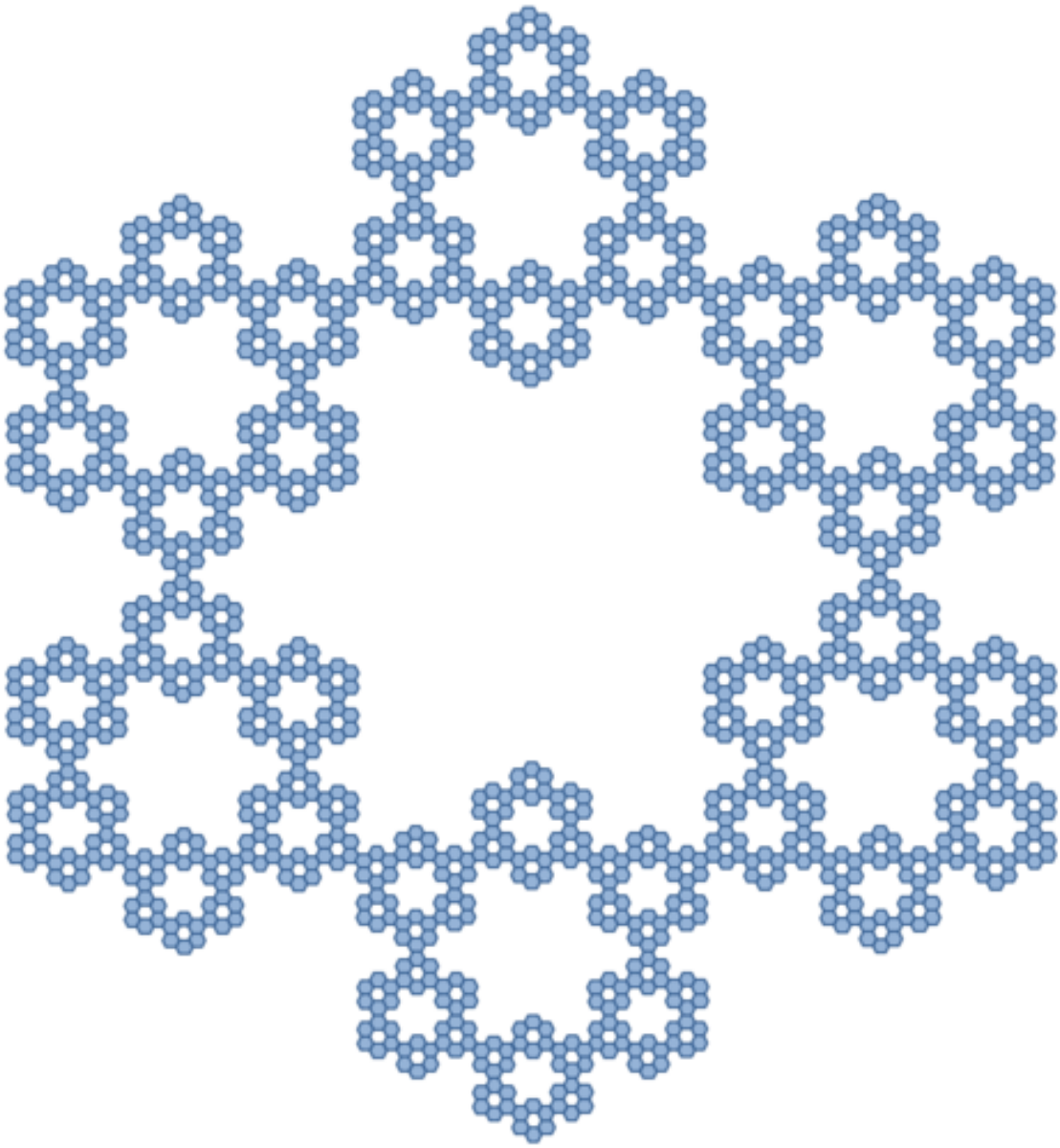

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Abstract

Using the hexagasket as a running example, we investigate Kigami's weak and pointwise formulations of a Laplacian Δ on certain post-critically finite self-similar fractal domains. We investigate how this relates to the probabilistic approach taken by other authors when defining processes on fractals. We consider several important PDEs including Laplace's equation, Poisson's equation and the linear heat equation on certain bounded fractal domains, and in the process explore harmonic structure, spectral theory and heat kernels. We obtain results about the non-existence, existence and regularity of the semilinear heat equations $\partial u/\partial t = \Delta u + f(x, u(x))$ on certain unbounded fractal domains, generalising results obtained by Falconer and Hu.

Contents

1 Introduction and motivation

Consider a natural and important physical question: how does heat diffuse on an object? For simplicity suppose the object is made of a uniform material. If the interior of the object looks like an open subset of $\mathbb{R}^n$ then, for physical reasons, the answer is given (to a good approximation) by a partial differential equation (PDE) known as the heat equation

$$\frac{\partial u}{\partial t}(t,x) = D\Delta u(t,x) \tag{1.1}$$

subject to appropriate initial and boundary conditions. Here, $u(t,x)$ is the temperature of the object at position x and time t, and D is a constant depending on the material that makes up the object. Δ is the Laplacian, a differential operator

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \ldots + \frac{\partial^2 f}{\partial x_n^2} \tag{1.2}$$

that measures, roughly, the rate at which the average value of a function over balls of small radius r deviates from the value of the function at that point as r grows. A physical derivation of (1.1) for the 1-dimensional case is given in [29]. There are several definitions of the Laplacian on 'classical' domains like open subsets of $\mathbb{R}^n$, including the pointwise definition (1.2) and the 'weak' formulation (discussed at the start of Section 3.2). All have one thing in common: they involve the *derivative* of a function, which is a limiting process that compares the value of a function with its values at nearby points in *all directions*. This implicitly assumes that it is possible to move a small distance in any direction inside the domain and stay inside the domain. This is certainly true on an open subset of $\mathbb{R}^n$, or more generally an open subset of a Riemannian manifold, but most objects in real life do not have such a nice smooth structure, and the initial physical question of how heat diffuses makes sense for 'rough' objects too. As it may be possible to guess from looking at some of the pictures shown in this dissertation, some fractals may model certain real-world materials better than any classical domain would, for example a porous material through which heat is flowing, or a catalyst for a chemical reaction that has a large surface area, or an antenna that is deliberately manufactured to have a self-similar structure so that it works well at many different frequencies. This motivates the idea of attempting to define analogues of the Laplacian on fractal domains and study the resulting 'PDEs' on these domains (noting that 'PDEs' is actually something of a misnomer because the operators on fractal domains cannot accurately be described as *differential* operators).

It must be noted, however, that true fractals do not exist in the real world because the fractality will inevitably break down at atomic scales, so objects in the real world can only display self-similarity (or, more commonly, statistical self-similarity) over a certain range of length scales. We will explore how to develop analogues on true fractals of the mathematical theory that is used to model physical processes on classical domains. The question of how closely the results of these analogous mathematical processes match the actual physical processes on real-world fractal-like objects is a difficult one, and not one that we will explore here; a thorough analysis of this question would require numerical simulations and/or physical experiments.

In Chapter 2, we review the necessary fractal geometry, with examples. In Chapter 3, we explore the notions of energy and harmonic functions on certain bounded post-critically finite self-similar fractal domains, and use this to define a weak formulation of the Laplacian, following the approach of Kigami [25]. We focus mostly on the hexagasket, carrying out the relevant analysis and calculations for the hexagasket in a similar way to how Strichartz did the relevant calculations for the Sierpinski gasket in [45]. We derive a pointwise formula for the Laplacian on the hexagasket and explore how this is related to the probabilistic definition of the Laplacian that has been given by other authors using random walks. In Chapter 4 we consider the solvability of two important and closely related PDEs, namely Laplace's equation $\Delta u = 0$ and Poisson's equation $\Delta u = f$. In the process, we deduce that the domain of the Laplacian is infinite-dimensional, and explore the structure of harmonic functions on the hexagasket. In Chapter 5 we explore the linear heat equation $\partial u/\partial t = \Delta u$ on bounded fractal domains using the spectrum of the Laplacian, before investigating the linear heat equation and heat kernels on unbounded fractal domains such as the unbounded hexagasket. In Chapter 6 we investigate the semilinear heat equation $\partial u/\partial t = \Delta u + f(x, u(x))$ on unbounded fractal domains for appropriate functions f, generalising results obtained by Falconer and Hu in [12] for the case $f(x, u(x)) = u(x)^p$. We give conditions for the existence of global solutions and prove results about the regularity of said solutions, and we give conditions for when blow-up will occur. We also consider how the conditions on f that lead to these different properties of the solution of the PDE depend on the geometrical and analytical properties of the structure of the unbounded domain.

We assume that the reader is familiar with standard concepts in analysis and measure theory such as the Lebesgue integral and L^p spaces. We assume familiarity with standard results about continuity of functions and convergence of sequences of functions and sequences of integrals of functions. We expect that the reader has some familiarity with the basics of fractal geometry, though we briefly review what is needed in Chapter 2. No knowledge about analysis on fractals is assumed.

2 Fractals

In this section we briefly review the main ideas in fractal geometry that are needed. As mentioned in the introduction, the definition of differential operators on fractal domains is significantly more complicated than on classical domains such as $\mathbb{R}^n$. Therefore, it is possible to make progress with only certain classes of fractals. The class of fractals best suited to development of the theory are post-critically finite sets, which are certain types of self-similar sets that we will explore below.

2.1 Self-similar sets

Much of the following material about self-similarity was put into a rigorous framework in Hutchinson's pioneering paper [23] and is discussed in Falconer's book [10].

Definition 2.1. An *iterated function system (IFS)* on a closed set $D \subset \mathbb{R}^n$ is a finite set of contractions on D, i.e. a finite set of maps $F_i : D \to D$ and numbers $r_i \in (0,1)$ such that $||F_i(x) - F_i(y)|| \leq r_i||x-y||$ for all $x, y \in D$.

Let H be the set of compact subsets of $\mathbb{R}^n$. It turns out that there is a metric on H called that Hausdorff metric that turns H into a complete metric space. Define $F : H \to H$ by $F(E) = \cup_i F_i(E)$. We then have the key theorem:

Theorem 2.1. *[Unique attractor] For an IFS $\{F_i\}$ on a closed set $D \subset \mathbb{R}^n$ with $||F_i(x) - F_i(y)|| \leq r_i||x-y||$ for all $x, y \in D$, there is a unique non-empty compact set $K \subseteq D$, called the* attractor *of the IFS, such that*

$$K = \bigcup_i K_i \qquad \text{where} \qquad K_i := F_i(K) \text{ for all } i.$$

Moreover, if E is a non-empty compact subset of D such that $F_i(E) \subseteq E$ for all i, then

$$K = \bigcap_{k=0}^{\infty} F^k(K).$$

Furthermore, K is the closure of the set of fixed points of finite compositions $F_{i_1} \circ \cdots \circ F_{i_p}$ of the F_i.

The above theorem can be proven directly or using Banach's contraction mapping theorem on the space of H.

Most of the sets that we will be considering will be attractors of an IFS that are self-similar:

Definition 2.2. A *self-similar set (SSS)* is the attractor of an IFS where all the contractions are similarities, i.e. $||F_i(x) - F_i(y)|| = r_i||x-y||$ for all x, y, where each $r_i \in (0,1)$.

We will see examples of such sets that satisfy an additional condition shortly. Now, we need some more notation to describe different specific particular locations within the fractal. The following definitions are related to the theory of analysis on fractals that was set up in Kigami's important paper [25] and illustrated for the example of the Sierpinski Gasket (which will be defined in Section 2.2) by Strichartz in [45].

Definition 2.3. A *word* $w = (w_1, \ldots, w_m)$ is a finite string of indices of the contractions. In this case, the *length* of the word is $|w| = m$. Write $K_w = F_w(K) = F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_m}$ for the *cell of level m corresponding to w*.

It is important to be careful about the order of composition of functions. Here, we of course use the convention in analysis that the function on the right is applied first.

When studying differential equations on fractals, if the fractal is not connected, then it makes sense to study the differential equation on the different connected components separately. Therefore we assume henceforth that all the fractals being considered are connected. However, many of the fractals will be 'close' to being disconnected in the following sense:

Definition 2.4. A (connected) SSS K is *finitely ramified* if, for any distinct words w and v of the same length, $K_w \cap K_v$ is finite.

Note that if V is the (finite) set of intersection points of pairs of distinct 1-cells, then $K\backslash V = \bigcup_i (K_i \backslash V)$ with the union disjoint, so $K\backslash V$ is not connected.

We now define an important class of finitely ramified sets called PCF fractals. Much of the theory of analysis of analysis on fractals has been done on PCF sets.

Definition 2.5. A connected SSS K is *post-critically finite (PCF)* if there is a finite subset V_0 of K called the *boundary* of K and whose elements are denoted by q_i for some $i \in \mathbb{N}$, such that if w and w' are distinct words of the same length, then

$$K_w \cap K_{w'} = F_w V_0 \cap F_{w'} V_0.$$

Some authors, such as Strichartz in [45], require each element of the boundary to be a fixed point of one of the self-similarities of the IFS, but we will not require this here. This definition is slightly different to the one first given by Kigami in [25]. It may seem odd that the term 'boundary' is used for a finite subset, but we will see that, perhaps surprisingly, in many situations it plays the role of the boundary of a classical domain in $\mathbb{R}^n$ when, for example, imposing Dirichlet or Neumann boundary conditions. It is immediate that $K\backslash V_1 = \bigcup_i (K_i \backslash V_1)$ with the union disjoint, so every PCF fractal is finitely ramified. The converse does not hold - an analogue of the Sierpinski gasket given in [2, p. 254] is finitely ramified but not PCF.

In order to define processes on K, it will be necessary to define processes on a sequence of graphs approximating K, which we now define inductively, based on [45, p. 92]. Let Γ_0 be the complete graph on V_0 (the complete graph is chosen so that the energy that we define in Chapter 3 will be nondegenerate). For $m \geq 1$ assume we have constructed the graph Γ_{m-1} on vertex set V_{m-1}. Define Γ_m to be the graph with vertex set

$$V_m = \bigcup_i F_i V_{m-1}$$

and such that $x \sim y$ (sometimes written $x \sim_m y$ to avoid ambiguity) if and only if there are x', y' adjacent in V_{m-1} such that $x = F_i x'$ and $y = F_i y'$ for some i. This is clearly equivalent to the existence of a word w of length m such that $x, y \in F_w V_0$, so V_m is the set of boundary points of m-cells. Since each element of the boundary is a fixed point of one of the maps F_i, it follows by induction that $V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots$, but the edge relations change; if $x \sim_m y$ then x and y generally will *not* be connected in Γ_{n+1}. Set

$$V_* := \bigcup_m V_m.$$

Since all the contraction ratios are strictly less than 1, given $x \in K$ and $\epsilon > 0$ there is $m \in \mathbb{N}$ sufficiently large such that all m-cells have diameter less than ϵ. An m-cell that x is an element of therefore has a boundary point distance less than ϵ from x, which shows that V_* is dense in K. This means that to define a continuous function on K it is sufficient to define its values on V_* which will be useful when later considering the pointwise definition of a Laplacian. We will need one more definition.

Definition 2.6. An element $x \in V_m \backslash V_0$ is a *junction point* if it belongs to more than one m-cell.

2.2 Examples

We now illustrate the content of Section 2.1 with the examples of the Sierpinski gasket (illustrated in Figure 1) and the hexagasket (illustrated in Figure 2). We will give two different IFSs that generate

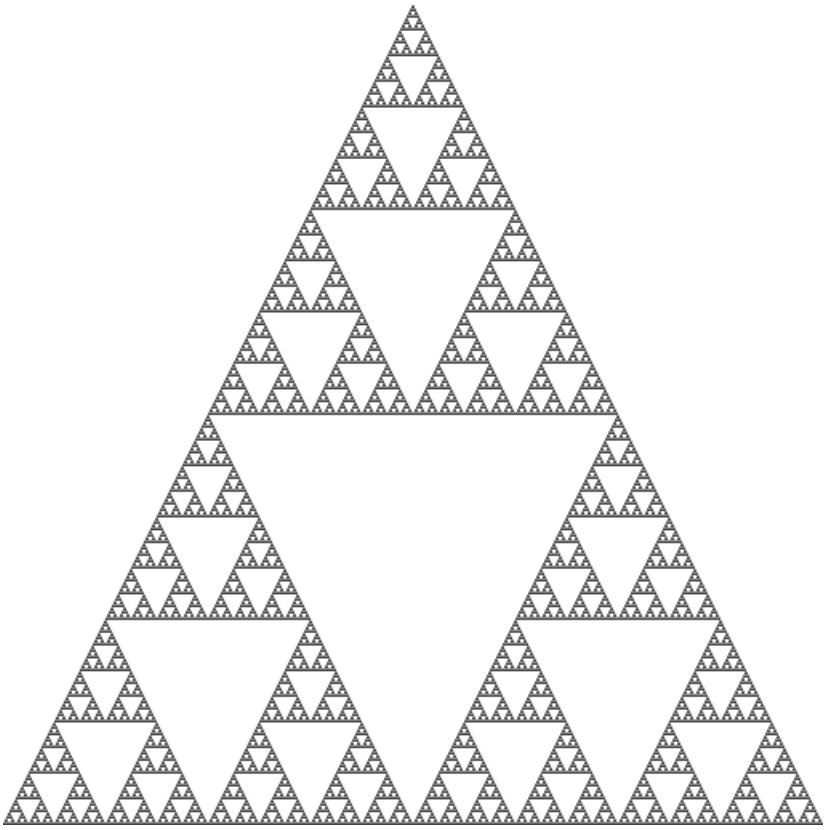

Figure 1: The Sierpinski gasket, from [32]

the hexagasket, and explore the differences between the two setups. The Sierpinski gasket is the simplest nontrivial example of a PCF SSS fractal, and was therefore one of the first on which the theory of differential equations was developed, see [5] for instance. It has received a disproportionate amount of attention in the literature, being used as the running example in [11, Section 12.4] and in [45], for example. Through much of his book [45], Strichartz cleverly uses one more example of a set, namely the unit interval $I = [0, 1]$, that is self-similar but not usually thought of as such as it is not a fractal. This is self-similar with the mappings $F_0(x) = x/2$ and $F_1(x) = (x + 1)/2$, and $V_m = \{k/2^m | 0 \leq k \leq 2^m\}$ so V_* is the dyadic rationals. The point of thinking of I in this way, which is much more complicated than one is used to, is because it is straightforward to do calculus on I; we already know that the Laplacian is given by the second derivative, for example. Therefore if we can formulate alternative definitions of familiar objects like the Laplacian in terms of the self-similar structure, then this can provide a useful way of constructing such objects on true fractals on which we cannot apply the usual methods of calculus like differentiation. Although some

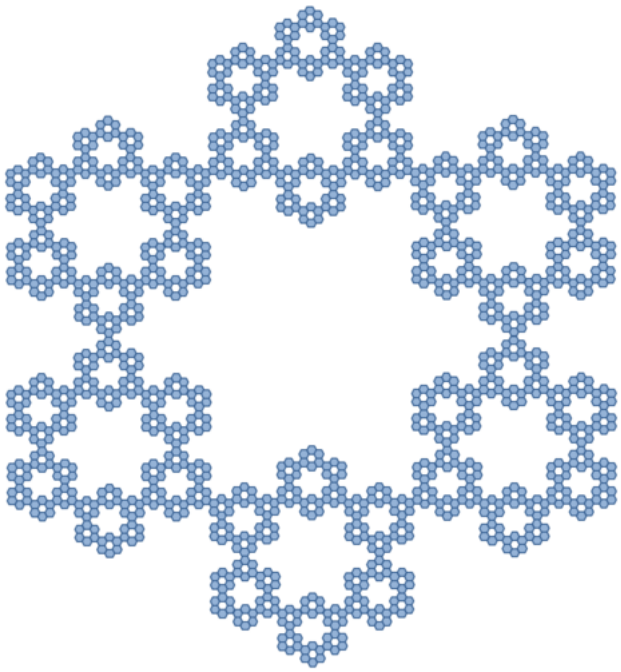

Figure 2: The hexagasket, from [39] (this is the same picture as on the title page)

of these alternative constructions on I (such as the insightful construction of the Green's function in [45, Section 2.6]) were in fact retrospective (the Green's function on the Sierpinski gasket had been constructed earlier by Kigami in [24]), they nonetheless provide the reader with motivation for what can otherwise feel like rather 'magical' constructions on PCF fractals. The hexagasket is less familiar than the Sierpinski gasket; much of the time we will use the hexagasket as the running example, precisely because fewer mathematicians choose to use it as such (and because it is mathematically and aesthetically pleasing to the author).

The Sierpinski gasket (or Sierpinski triangle) is the invariant set of the following IFS. Pick three non-collinear points q_0, q_1, q_2 in the plane and define, for $i = 1, 2, 3$,

$$F_i x = \frac{1}{2}(x - q_i) + q_i,$$

which clearly fixes q_i. By considering geometrically the locations of K_w for different words w of length m, it is clear that the Sierpinski gasket is PCF with boundary $\{q_i, q_2, q_3\}$ - for example, $K_1 \cap K_2 = \{F_1 q_2\} = \{F_2 q_1\}$. Note that the contraction ratios are $1/2$. Note also that the precise relative locations of the points q_i do not change the topological structure of the resulting invariant set, so the choice of points will only matter for some applications. In diagrams of the Sierpinski gasket, the triangle formed by the points is usually equilateral (as in Figure 1), but sometimes may be right-angled. By Theorems 2.4 and 2.3, the Hausdorff dimension of the Sierpinski gasket is the value $s \geq 0$ such that

$$\left(\frac{1}{2}\right)^s + \left(\frac{1}{2}\right)^s + \left(\frac{1}{2}\right)^s = 1,$$

or equivalently $2^s = 3$, so

$$s = \log_2 3 = \frac{\log 3}{\log 2} \approx 1.58.$$

The hexagasket, also known as the Sierpinski hexagon or hexakun, is illustrated in Figure 2. We first define the more obvious but (as we will see) mathematically less natural IFS. Pick six points $\{q_1, \dots q_6\}$ in a regular hexagon and define six similarities $F_1, \dots, F_6$, each with contraction ratio $1/3$ and each fixing a different boundary vertex. Say F_i fixes q_i, so $F_i(x) = (x - q_i)/3 + q_i$. Again, this is PCF with boundary $\{q_1, \dots q_6\}$; for example, $K_1 \cap K_2 = \{F_1 V_3\} = \{F_2 V_5\}$. Denote the attractor by H'; it has D_6 symmetry. By Theorems 2.4 and 2.3, the Hausdorff dimension is the value $s \geq 0$ such that

$$\left(\frac{1}{3}\right)^s + \left(\frac{1}{3}\right)^s + \left(\frac{1}{3}\right)^s + \left(\frac{1}{3}\right)^s + \left(\frac{1}{3}\right)^s + \left(\frac{1}{3}\right)^s = 1,$$

or equivalently $3^s = 6$, so

$$s = \log_3 6 = \frac{\log 6}{\log 3} \approx 1.63.$$

For the Sierpinski gasket, all non-boundary vertices of the graphs Γ_m are junction points, but it can be seen from Figure 3 that for the hexagasket, $F_2 q_1 \in \Gamma_1$, for example, is not a junction vertex because the only 1-cell it is contained in is K_2.

The second way of setting up the hexagasket again starts with six points $\{q_1, \dots q_6\}$ in a regular hexagon; now it will be convenient for the centre of the hexagon to be the origin in $\mathbb{R}^2$. The new IFS consists of six similarities $F_1, \dots, F_6$, with F_j now defined by $F_j(x) = R_{j-1}(x/3) + 2q_j/3$, where R_j is a clockwise rotation by angle $j\pi/3$ about the origin. Again this is PCF, but this time with boundary consisting of the three vertices $\{q_1, q_3, q_5\}$ (every second vertex of the regular hexagon) - for example, $K_1 \cap K_2 = \{F_1 q_3\} = \{F_2 q_5\}$. Denote the attractor by H; this has D_3 symmetry. Now, by the D_6 symmetry of H', it is clear that H' is invariant under the second IFS. Therefore by uniqueness of the attractor in Theorem 2.1, we must have $H = H'$ as sets.

The SSS considered so far are clearly bounded. It will sometimes be of interest to consider unbounded sets, by starting with a SSS as above and inductively extending outwards by similarity. Strichartz gives a general theory of such 'blowups' in [46], but we illustrate this concept with an example, namely the hexagasket H. Define $H_0 = H$ and choose a boundary vertex q. Extend the two sides of the hexagon that intersect at q by a factor of three. Let H_1 be the hexagasket generated from the regular hexagon that has q and the other endpoint of each of these two extended lines as vertices. Inductively define $H_3, H_4, \dots$ so that H_n is a hexagasket that is three times bigger than H_{n-1}, and $H_0 \subset H_1 \subset H_2 \subset \dots$. Note that each H_j is a scaled copy of H. Define the unbounded hexagasket to be the set

$$\bigcup_{i=0}^{\infty} H_i, \tag{2.1}$$

part of which is shown in Figure 4.

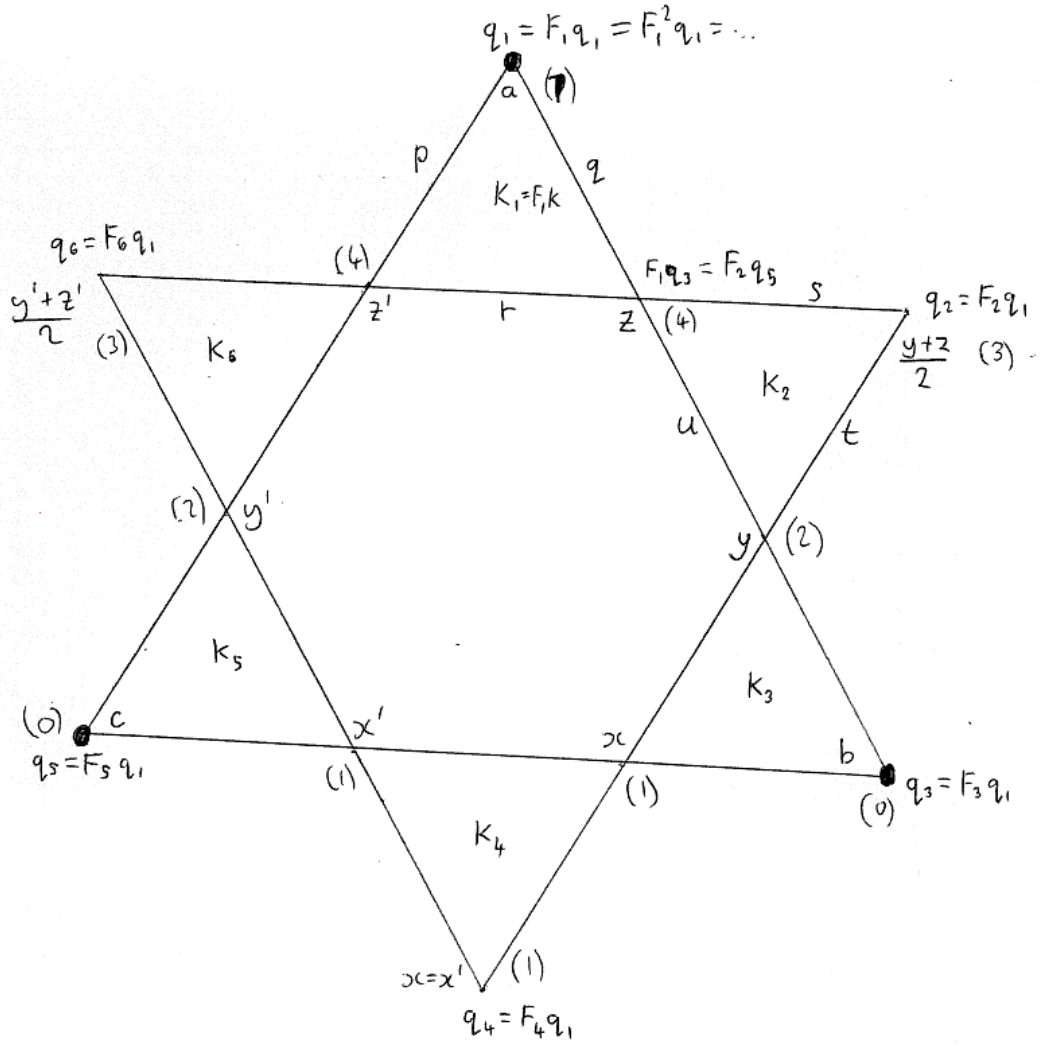

Figure 3: Γ_1 for the hexagasket H

2.3 Measures

Due to the difficulty in defining a 'derivative' on fractals that we have already discussed, we will see that to solve a 'differential equation' on a fractal domain one must usually consider weak solutions of the corresponding integral equation, since this formulation of a differential equation on $\mathbb{R}^n$ often has a more natural analogue on fractal domains. But integration requires a measure, so we next consider natural measures on fractals. Is will be convenient to work with probability measures (whose total mass is 1) that satisfy certain conditions that make it what Strichartz [45, p. 5] calls a *regular* probability measure:

Definition 2.7. A *regular probability measure* on a PCF SSS K is a function μ that assigns weights $\mu(C)$ to all cells C in such a way that the following four conditions hold:

- Positivity: $\mu(C) > 0$

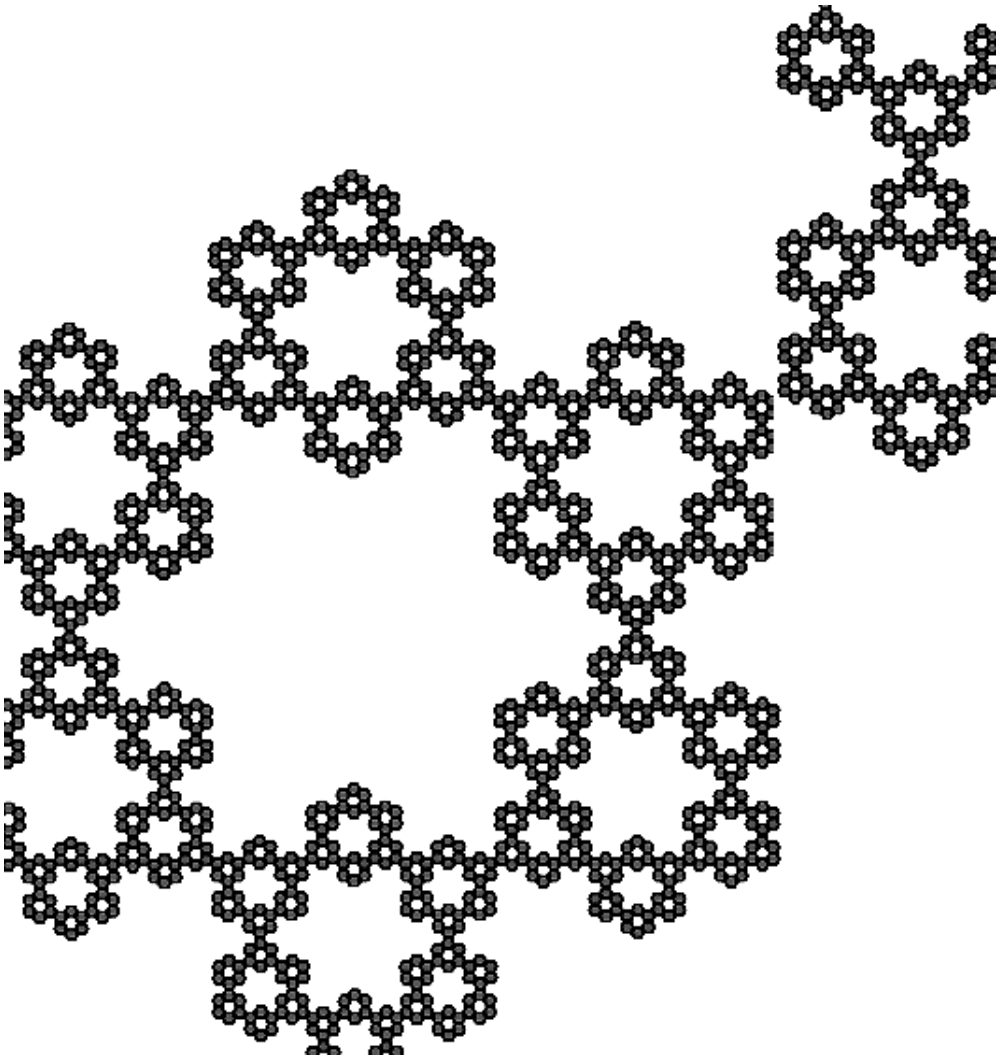

Figure 4: Part of the unbounded hexagasket, made by the author using the picture [39]

- Additivity: If $C = \bigcup_{j=1}^{N} C_j$ and the cells C_j are disjoint or intersect only at boundary points, then $\mu(C) = \sum_{j=1}^{N} \mu(C_j)$.

- Continuity: $\mu(C) \to 0$ as the size of C tends to 0, i.e. as the length of the word corresponding to C tends to ∞.

- Probability: $\mu(K) = 1$.

Note that μ is not yet a measure because the set of cells does not form a σ-algebra. We may, however, use the fact that the additivity condition implies that points have zero measure to extend μ to a pre-measure on the algebra of finite unions of cells, as follows. If $A = \bigcup_{j=1}^{N} C_j$ is a finite union of cells that are disjoint except for possible point intersections, then define

$$\mu(A) = \sum_{j=1}^{N} \mu(C_j). \tag{2.2}$$

There is clearly a unique canonical decomposition of A into maximal cells, i.e. cells that are not

contained in any larger cell that is also a subset of A. Then any decomposition of A into a finite union of cells is some subdivision of this canonical decomposition, and the additivity condition ensures that measure is preserved, so (2.2) is well-defined.

Now, the total measure is 1, which is finite, so the pre-measure μ is certainly σ-finite, so Carathéodory's extension theorem guarantees that μ extends to a unique measure on the σ-algebra generated by this algebra.

Instead of the additivity condition, it suffices to check the case when the cell C is being decomposed into cells of the next level, i.e. that for each word w we have

$$\mu(F_w K) = \sum_i \mu(F_w F_i K). \tag{2.3}$$

If this holds, then given an arbitrary subdivision, we may apply this subdivision as many times as is required to derive the additivity property for the arbitrary subdivision. Therefore the regular probability measures on K are precisely those obtained by the following procedure. The whole set K must have weight 1, and then, assuming inductively that we have assigned weights to all the m-cells, for each m-cell C, assign positive weights to each of the $(m+1)$-cells that C is composed of in such a way that (2.3) is satisfied and the continuity assumption holds in the limit.

This shows that there are very many possible regular probability measures. A simple class of such measures, called *self-similar* measures, are obtained by assigning weights in the same way for each cell and each level. More precisely, for each similarity F_i, choose a weight $\mu_i > 0$ in such a way that the μ_i sum to 1; these will be the weights used for the splitting. Then if $|w| = m$, define

$$\mu_w := \mu(F_w K) = \prod_{j=1}^{m} \mu_{w_j}. \tag{2.4}$$

This is equivalent to the following relation:

$$\mu(A) = \sum_i \mu(A \cap F_i K) \tag{2.5}$$

Indeed, (2.4) implies that (2.5) certainly holds for cells and unions of cells that are disjoint except possible at boundary points, so (2.5) must holds for all measurable sets. Conversely, (2.5) can be applied inductively to derive (2.4). Another useful identity is the following:

Proposition 2.2 (Self-similar identity)**.** *For a self-similar measure with notation as above, and any (measurable) set A, we have*

$$\mu(A) = \sum_i \mu_i \mu(F_i^{-1} A).$$

Proof. From [45, page 6]. Note that $F_i^{-1}A = F_i^{-1}(A \cap F_i K)$ and $A \cap F_i K = F_i F_i^{-1}(A \cap F_i K)$ so (2.4) gives that $\mu(A \cap F_i K) = \mu_i \mu(F_i^{-1}(A \cap F_i K))$. Therefore $\mu(A \cap F_i K) = \mu_i \mu(F_i^{-1} A)$, and substituting this into (2.5) gives

$$\mu(A) = \sum_i \mu_i \mu(F_i^{-1} A),$$

as required. □

This identity in fact determines the regular probability measure μ. Indeed, fixing a particular j and replacing A by $F_j A$ in the self-similar identity, together with the condition that the total mass is 1, shows (2.4) by induction.

Recall some familiar properties of the Lebesgue measure that will in fact apply to most of the measures considered on fractal domains too. In the following definitions, let X be a topological space with a σ-algebra Σ and let μ be a measure on (X, Σ).

Definition 2.8. A measurable subset A of X is *inner regular* if

$$\mu(A) = \sup \{\mu(F) | F \subseteq A, F \text{ compact and measurable}\}.$$

Definition 2.9. A measurable subset A of X is *outer regular* if

$$\mu(A) = \inf \{\mu(F) | G \supseteq A, G \text{ open and measurable }\}.$$

Most of the measures encountered here will be Radon measures:

Definition 2.10. A measure μ on the σ-algebra of Borel sets of a Hausdorff topological space is a *Radon measure* if it is finite on all compact sets, outer regular (on all measurable/Borel sets), and inner regular on all open sets.

Intuitively this means that the measure is in some sense compatible with the topology of the space. If the space X is in addition locally compact and $K(X)$ is the space of continuous real-valued functions with compact support on X, then $K(X)$ can be given a natural topology. With respect to this topology, the positive linear map $K(X) \to \mathbb{R}, f \mapsto \int_X f d\mu$ is *continuous* if μ is a Radon measure. Recall that the Hausdorff separation condition asserts that distinct points have disjoint neighbourhoods.

There is a general class of objects on which the analysis of PDEs has generated much interest recently, namely metric measure spaces.

Definition 2.11. Let (M, d) be a metric space that is locally compact with the induced topology and is separable, i.e. has a countable dense subset. Let μ be a Radon measure on M and assume that μ has *full support*, in other words that all open neighbourhoods have positive measure (intuitively this means that all parts of the set are 'seen' by the measure). Then the triple (M, d, μ) is known as a *metric measure space*.

There are many technical conditions in this definition, but it essentially means that the space is a metric space whose topology is reasonable, with a measure that is compatible with the topology and sees the entire set.

2.4 Hausdorff dimension

One of the most important notions in fractal geometry is that of dimension, as a measure of the size of a set that should coincide with the usual notion of integer dimension for smooth manifolds. There are many different notions of dimension, and later we will see that different notions of dimension are

important in the study of differential equations on fractal domains, in particular when obtaining bounds for functions called heat kernels. One of the most important types of dimension is the Hausdorff dimension which can be defined using the Hausdorff measure, which requires several definitions.

Definition 2.12. The *diameter* of a set $U \subseteq \mathbb{R}^n$ is

$$|U| = \sup\big\{||x-y|| \big| x, y \in U\big\}.$$

Definition 2.13. Let $F \subseteq \mathbb{R}^n$ and $\delta > 0$. A *δ-cover* of F is a set of subsets $\{U_i\}$ of $\mathbb{R}^n$ such that $0 \leq |U_i| \leq \delta$ for each i, that *cover* F, i.e.

$$F \subseteq \bigcup_i U_i.$$

Definition 2.14. For $s \geq 0$ define

$$H^s_\delta(F) = \inf\left\{ \sum_{i=1}^{\infty} |U_i|^s \,\middle|\, \{U_i\} \text{ is a countable } \delta\text{-cover of } F \right\}.$$

Here we use the convention that every finite set is countable. As δ decreases, the class of δ-covers of F is reduced, so the infemum increases (though perhaps not strictly), so converges to a limit

$$H^s_\delta(F) \to H^s(F) \in [0, \infty] \text{ as } \delta \to 0^+.$$

The limit is the *s-dimensional Hausdorff measure* of F. It can be shown, for example in [7], that this is an outer measure on $\mathbb{R}^n$, and so its restriction to the H^s-measurable sets is a measure, as is its further restriction to the Borel sets.

Definition 2.15. It is straightforward to see (as in [10, pp. 47-8]) that for any $F \subseteq \mathbb{R}^n$ there is a unique $s \geq 0$ such that if $0 \leq t < s$ then $H^t(F) = \infty$ and if $t > s$ then $H^t(F) = 0$. This value s is known as the *Hausdorff dimension* of F. The s-dimensional Hausdorff measure of F may be any value in $[0, \infty]$; if it is positive and finite then F is called an *s-set*.

To obtain a nice result about the dimension of self-similar sets, we need a technical condition that says, intuitively, that the components K_i do not overlap 'too much':

Definition 2.16. An IFS satisfies the *open set condition (OSC)* if there is a non-empty, bounded, open set $V \subset \mathbb{R}^n$ such that $V \supseteq \bigcup_i F_i(V)$ with the union disjoint.

The next theorem was one of Hutchinson's key achievements in [23].

Theorem 2.3. *[Hausdorff dimension of SSS] With notation as above, if an IFS satisfies the open set condition then the Hausdorff dimension of the attractor is equal to the value $s \geq 0$ that is given by*

$$\sum_i r_i^s = 1.$$

Moreover, for this value of s, the fractal has positive and finite s-dimensional Hausdorff measure.

It is well known that if f is a similarity transformation with scale factor $\lambda > 0$ then for any measurable set F we have the scaling property $H^s(f(F)) = \lambda^s H^s(F)$. This means that for any SSS that satisfies the OSC, the scaled version of Hausdorff measure $B \mapsto H^s(B)/H^s(F)$ satisfies (2.4) with $\mu_i := r_i^s$ for all measurable subsets $B \subseteq F$. We have seen that (2.4) is equivalent to (2.5), and determines the measure to be the unique self-similar measure with weights r_i^s. Note that these weights sum to 1 by Theorem 2.3, so this scaled Hausdorff measure is indeed a probability measure. If μ is used without qualification on a SSS that satisfies the OSC, then this is the measure that μ will be denoting, and it is called the *standard (probability) measure* on F. The measure μ (in fact any constant multiple of μ) is an *s-Ahlfors regular* measure, and in particular it satisfies the condition

$$\exists m > 0 \quad \forall r > 0 \, \forall x \in G \quad \mu(B_r(x)) \leq m r^s. \qquad (*)$$

The proof of [10, Theorem 9.3] contains the ingredients required to prove this. The condition $(*)$ is also satisfied by any constant multiple of the restriction of the s-dimensional Hausdorff measure to the 'blowup' of a SSS like the hexagasket of Hausdorff dimension s.

It is conjectured that every connected, finitely-ramified set satisfies the OSC, but this is still open at the time of writing. Partial results cover many of the cases of interest here. The next theorem was proven in [3, p. 1463]:

Theorem 2.4. *Every connected, planar, finitely-ramified SSS satisfies the open set condition.*

In particular, every connected planar PCF SSS (like the Sierpinski gasket and hexagasket) satisfies the OSC.

The class of nested fractals is useful for proving heat kernel bounds (to be defined later). Defining this class will require one further definition:

Definition 2.17. A fixed point q of a similarity is an *essential fixed point* if there is a fixed point q' of a different similarity and distinct i, j such that $F_i(q) = F_i(q')$.

Definition 2.18. A SSS K is a *nested fractal* if it satisfies the OSC, is PCF with boundary $V_0 = \{\text{essential fixed points}\}$, and satisfies the following two additional conditions:

(Connectivity) For any 1-cells C and C' there is a sequence $C_1, \ldots, C_n$ of 1-cells such that $C = C_1$, $C' = C_n$ and for $1 \leq i < n$, C_i and C_{i+1} have a point in common.

(Symmetry) For distinct essential fixed points q and q', the reflection in the hyperplane of points equidistant from q and q' maps V_n to itself for all $n \geq 0$.

This definition is from Lindstrøm's paper [31]. Clearly the Sierpinski gasket and hexagasket are nested fractals.

In the discussion of the possible approaches to defining a Laplacian, we will mention the notion of a curve, as a subset of the fractal domain under consideration.

Definition 2.19. A *rectifiable curve* is a continuous injection $[a, b] \to \mathbb{R}^n$ where $a, b \in \mathbb{R}, a < b$, whose *length*

$$L(C) := \sup \sum_{i=1}^{m} ||x_i - x_{i-1}||$$

is positive and finite, where the supremum is taken over all finite dissections of the curve by points $x_0, x_1, \ldots, x_m$ in that order along the curve.

The following intuitive proposition is proven in [10, p. 89]:

Proposition 2.5. *The length of any rectifiable curve is equal to its one-dimensional Hausdorff measure.*

3 The Laplacian Δ

3.1 Motivation and overview

On open subsets of $\mathbb{R}^n$, the Laplacian is a second-order differential operator that measures, in some sense, the rate at which the average value of a function over small balls of small radius r deviates from the value of the function at that point as r grows. The operator Δ is also the flux density of the gradient of a scalar function. It is extremely important in physics because many different physical processes are modelled by differential equations that involve the Laplacian. For example, diffusion processes are modelled by the diffusion equation, which involves Δ (see Chapter 5 for more details regarding the simplest diffusion equation, namely the heat equation). The operator Δ is also used extensively in Maxwell's equations of electromagnetism. Harmonic functions, i.e. solutions of Laplace's equation $\Delta u = 0$, represent the possible gravitational fields in free space, and in the absence of charge, the electric potential is a harmonic function. The Laplacian even appears in the wave equation, Schrödinger's equation in quantum mechanics, and in equations relating to image processing.

This suggests that in order to study physical processes on domains that are more complicated than $\mathbb{R}^n$, it is sensible to try to construct operator(s) on such domains that also measure the rate of deviation of the average of a function over small balls, and that share many of the properties of the classical Laplacian. The most obvious idea when trying to do this is to try to adapt the usual definition (1.2) of the Laplacian on $\mathbb{R}^n$. This is possible on Riemannian manifolds because they have a differentiable structure; the Laplace-Beltrami operator was developed in the 19th Century as an analogue of the Laplacian on Riemannian manifolds. Much more recently, there has been a significant amount of work done to develop analogues of the Laplacian on *fractal* domains, which are the topic of interest here. On such rough domains, adapting (1.2) is much harder. It is of course unlikely that an arbitrary point on a fractal subset of $\mathbb{R}^n$ will lie in the interior of line segments inside the fractal that are parallel to the coordinate axes, so the definition cannot be used directly. However, from the pictures it is clear that in many fractals there will be many line segments, so an idea would be to use the restriction of a function to those line segments to compute certain ordinary directional derivatives, and then piece them together in some clever way to define a Laplacian. However, this turns out to be very difficult. In any case, as noted in the introduction of [45, p. xi], a generic point on even the simplest nontrivial fractal on which these methods can be applied, namely the Sierpinski Gasket, does not lie on any line segments. Strichartz notes further that more general fractals to which these methods can be applied "do not necessarily contain any rectifiable curves."

The first successful method to define a Laplacian on fractal domains was the probabilistic method. A Markov process, analogous to Brownian motion on classical domains, is defined on certain fractals (this is highly nontrivial), and a Laplacian is then defined as an 'infinitesimal generator' of this process. This was explored in the 1980s by authors such as Goldstein, Kusuoka, Barlow, Perkins and Lindstrøm, see [4, 5, 18, 27, 31]. We will explore this approach in Section 3.5. The other method is based on calculus, and involves defining discrete operations on a sequence of graphs that approximate the fractal, before taking a limit to define the required operations on the fractal itself. As we will see, this method can be used to formulate the Laplacian in either a weak or pointwise sense. This theory was developed by Kigami in [25], and discussed in a more accessible way and in less generality by Strichartz in [45]. Many mathematics books and articles on the subject of analysis on fractals will use either one of these two perspectives, without drawing them together. In this dissertation we discuss both perspectives, mostly using the example of the hexagasket. Currently, a Laplacian has only been constructed on certain specific classes of fractals, for example many PCF fractals. In this chapter, we first define a notion of 'energy' on the hexagasket, and use this to define harmonic functions. We then use the energy to define a weak formulation of the Laplacian, and give a pointwise formula for it. Finally, we show how to recover the definition of the Laplacian using random walks.

3.2 Energy and harmonic functions

The first form of the Laplacian we will construct is the weak formulation. Although it is less familiar than the pointwise definition (1.2) on classical domains, it is in some ways more natural than the pointwise definition both on classical and on fractal domains. We first recall the weak formulation on an open, bounded subset $U \subset \mathbb{R}^n$. By Green's identity, which is essentially the higher dimensional equivalent of integration by parts, if u is twice differentiable and v is differentiable and v vanishes on the boundary of U, then

$$\int_U (\Delta u)v = -\int_U \nabla u \cdot \nabla v.$$

This suggests that it may be sensible to define a 'weak' version of Δu to be the function f such that

$$\int_U fv = -\int_U \nabla u \cdot \nabla v \tag{3.1}$$

for all differentiable functions v vanishing on the boundary of U. An advantage of this is that this definition makes sense even when U is not twice differentiable. This approach can be used to solve Poisson's equation $-\Delta u = g$, for instance. Indeed, if u is a twice differentiable solution of this equation then, integrating against any differentiable v that vanishes on the boundary of U, we have

$$\int_U gv = -\int_U (\Delta u)v = \int_U \nabla u \cdot \nabla v$$

which is known as the weak formulation of Poisson's equation. The advantage is that now we can look for a solution u in a suitable Hilbert space (in this case the Sobolev space $H_0^1(U)$) to find a weak solution, using the fact that the map $(u, v) \mapsto \int_U \nabla u \cdot \nabla v$ (known as *energy*) is a symmetric

bilinear form on this space; this process allows the tools of linear algebra to be used to tackle PDE problem. Once we have such a candidate solution, we can then try to show that this solution is sufficiently regular, and then show that that would mean that it is in fact a solution of the original strong form of Poisson's equation. A description of how this can be done is in [35]. The quadratic form $u \mapsto \int_U ||\nabla u||^2$ corresponding to the energy bilinear form, known as the *(Dirichlet) energy* of u, measures how variable the function u is. For physical reasons (see [14]), it is a constant multiple of the energy due to the electric field when the electric potential is u, which explains the name 'energy.'

At first it may not be clear how this is helpful for the case of fractal domains, since the energy has also been defined in terms of derivatives, and we cannot differentiate on fractal domains. In fact it is easier to define a notion of energy for suitable functions on PCF fractals using graph approximations to the fractal, a common method in analysis on PCF fractals. We follow the setup of Strichartz [45, Section 1.3]. We define the energy of a real-valued function u on the vertices of a finite, connected graph G to be

$$E_G(u) = \sum_{x \sim y} (u(x) - u(y))^2. \tag{3.2}$$

The sum is over all edges; if it were over all ordered pairs of vertices (x, y), then this would sum over each edge twice and we would have to compensate by multiplying the whole expression by a factor $1/2$. We will see another reason why 'energy' is an appropriate name at the end of Section 4.1. Clearly $E_G(u)$ is a real quadratic form in u, so the polarisation identity gives the associated bilinear form

$$E_G(u, v) = \frac{1}{4}(E_G(u + v) - E_G(u - v)) = \sum_{x \sim y} (u(x) - u(y))(v(x) - v(y)), \tag{3.3}$$

which is clearly symmetric in u and v and satisfies $E_G(u, u) = E_G(u)$. Clearly $E_G(u)$ is constant if and only if u is constant on every connected component of G, so since G is connected this is equivalent to u being constant on G. Therefore E_G is positive definite on the quotient space of functions on the vertex set of G modulo the subspace of constant functions, and therefore gives an inner product on this space.

The graphs we will work with are the Γ_i defined in Section 2.1, which satisfy $V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots$, so we consider the situation where G and G' are graphs on vertex sets V and V' respectively, with $V \subset V'$. For a function u defined on V, we seek to minimise $E_{G'}(u)$ among all extensions of u to V'; this is equivalent to minimising a continuous function on $\mathbb{R}^{|V'|-|V|}$. If the extension takes values on $V' \backslash V$ that are very large in modulus compared with the values on V, then the energy is certainly not minimised, so this is actually a minimisation problem on a compact subset of $\mathbb{R}^{|V'|-|V|}$, and therefore has at least one solution $\tilde{u}$, called a *harmonic extension* of u. Any such extension will have each of the $|V'| - |V|$ partial derivatives equal to zero. If x is the value of extensions u' of u at some particular vertex in $V' \backslash V$ then treating x as a variable and computing the partial derivative of $E_{G'}(u')$ with respect to x gives

$$\frac{\partial E_{G'}(u')}{\partial x} = \sum_{x \sim y} 2(u(x) - u(y)),$$

so the condition $\frac{\partial E_{G'}(u')}{\partial x} = 0$ is simply saying that the value of u' at x is the average of its neighbours. In most cases of interest, $\tilde{u}$ will be the *unique* energy-minimising extension of u (meaning that if $\tilde{u}|_V = u$ and if u' is also such that $u'|_V = u$ then $E_{G'}(\tilde{u}) \leq E_{G'}(u')$ with equality if and only if $u = \tilde{u}$).

Now, in many, but not all, PCF fractals it will happen that there is an $r \in (0, 1)$ such that for all $m \geq 0$, when $G = \Gamma_m$ and $G' = \Gamma_{m+1}$, for *all* functions u on V we have the *renormalisation equation*

$$E_{G'}(\tilde{u}) = rE_G(u).$$

This means that if we renormalise the definition of energy on G' by multiplying by $1/r$ (defining $\mathcal{E}_G(u) := r^{-1}E_G(u)$) then energy increases with extension from G to G' except in the case of a harmonic extension, when it remains unchanged. Minimising $\mathcal{E}_G$ is clearly the same problem as minimising E_G, with the same function achieving the minimum. Strichartz shows in [45, Section 1.3] that this works for the unit interval and the Sierpinski gasket.

We now show that this also works for the hexagasket H with small boundary (of size three). In [45, pp. 102-4] Strichartz touches on this but omits the calculation of the energy of the harmonic extension of a general function on the boundary; we do this calculation and, to compute the harmonic extension algorithm, use an argument that exploits the symmetry of the hexagasket and yields a simpler computation. We start by finding the harmonic extension from Γ_0 to Γ_1 of the function that takes the value 7 on one of the boundary vertices (q_1 in Figure 3) and 0 on the other two boundary vertices (q_3 and q_5); it will become clear during the course of the computation why the particular choice of '7' happens to simplify the algebra. If a harmonic extension takes values z, y, x, x', y', z' on the vertices in Figure 3, then by symmetry $x = x'$, $y = y'$ and $z = z'$. At each of the three non-boundary vertices of the hexagon, the value is the average of the value of the two neighbours, so for example the value is $(y + z)/2$ at q_2.

As an aside, note that before even doing any calculations, it is already clear that any harmonic extension will be non-constant but will be constant on one of the 1-cells. When we later define harmonic functions on K in analogy with harmonic functions on connected, bounded, open subsets of $\mathbb{R}^n$, it will be clear that this means that there is a non-constant harmonic function that is constant on a 1-cell. Constant functions will also be harmonic, and will obviously be constant on the same 1-cell. This is utterly different to the case of harmonic functions on open connected subsets $U \subset \mathbb{R}^n$, which are real analytic functions and hence (as follows from a remark by Lee in [30, p. 46]) determined by their values on any open subset of U. It is also different to the case of the Sierpinski gasket that Strichartz analyses on [45, pp. 13-17]. This is a good example of how the geometry of the fractal domain can result in very different behaviour from what might be expected. We will return to this in Section 4.1.

Returning to the calculation of the harmonic extension, we see that the mean value condition

gives three linear equations in the three unknowns x, y and z, namely

$$\begin{aligned} 4x &= x + x + 0 + y \\ 4y &= x + 0 + z + \frac{y+z}{2} \\ 4z &= 7 + z + y + \frac{y+z}{2} \end{aligned}$$

or equivalently

$$\begin{aligned} x &= y/2 \\ \frac{7y}{2} &= x + \frac{3z}{2} \\ \frac{5z}{2} &= \frac{3y}{2} + 7 \end{aligned}$$

Substituting the first equation into the second turns the second into $2y = z$. Substituting this into the third equation gives $y = 2$ which implies $z = 4$ and $x = 1$. These values do indeed solve the three original equation, and are therefore the unique solution.

By symmetry, the same the solution would have the same structure if the '7' were at either of the other boundary vertices. As we have already noted, the energy function that we are minimising is quadratic, so the minimising equations are linear. Therefore if the initial values on the boundary are a, b, c as in Figure 3, then the solution is

$$z = \frac{4}{7}a + \frac{2}{7}b + \frac{1}{7}c \tag{3.4}$$

and so on. Formally this is the result of a solution of the matrix equation

$$M \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

where M is the same matrix which gave $(x, y, z)^T = (1, 2, 4)^T$ as the solution of

$$M \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 0 \end{pmatrix}. \tag{3.5}$$

Since we have seen that $(x, y, z)^T = (1, 2, 4)^T$ was the *unique* solution of (3.5), M must be invertible, so the solution given by (3.4) is unique, and therefore gives the unique harmonic extension of a function on the boundary. The value at the vertex adjacent to z and y should by symmetry depend on a and b in a similar way, and indeed it is

$$\frac{1}{2}\left(\frac{4}{7}a + \frac{2}{7}b + \frac{1}{7}c + \frac{2}{7}a + \frac{4}{7}b + \frac{1}{7}c\right) = \frac{3}{7}a + \frac{3}{7}b + \frac{1}{7}c, \tag{3.6}$$

We now compute the energy of an arbitrary harmonic extension, using (3.4) and (3.6). Edge q gives energy

$$\left(a - \left(\frac{4}{7}a + \frac{2}{7}b + \frac{1}{7}c\right)\right)^2 = \left(\frac{3}{7}a - \frac{2}{7}b - \frac{1}{7}c\right)^2$$

and similarly edge p gives energy $\left(\frac{3}{7}a - \frac{1}{7}b - \frac{2}{7}c\right)^2$. Edge s gives energy

$$\left(\left(\frac{4}{7}a + \frac{2}{7}b + \frac{1}{7}c\right) - \left(\frac{3}{7}a + \frac{3}{7}b + \frac{1}{7}c\right)\right)^2 = \left(\frac{1}{7}a - \frac{1}{7}b\right)^2$$

and similarly edge t gives energy $\left(\frac{1}{7}b - \frac{1}{7}a\right)^2$. Edge r gives energy

$$\left(\left(\frac{4}{7}a + \frac{2}{7}b + \frac{1}{7}c\right) - \left(\frac{4}{7}a + \frac{1}{7}b + \frac{2}{7}c\right)\right)^2 = \left(\frac{1}{7}b - \frac{1}{7}c\right)^2.$$

Edge u gives energy

$$\left(\left(\frac{4}{7}a + \frac{2}{7}b + \frac{1}{7}c\right) - \left(\frac{2}{7}a + \frac{4}{7}b + \frac{1}{7}c\right)\right)^2 = \left(\frac{2}{7}a - \frac{2}{7}b\right)^2.$$

The energies corresponding to the remaining edges are precisely those expressions of the same form as above with a cyclic permutation of $\{a, b, c\}$. Considering the term a^2 we see that the six edges of the same form as p and q contribute a term with coefficient $\frac{3^2+3^2+2^2+2^2+1^2+1^2}{7^2} = \frac{28}{49}$. Edges of the same form as s and t contribute $\frac{1+1+1+1}{49} = \frac{4}{49}$. The three edges of the form r contribute $\frac{1+1}{49} = \frac{2}{49}$. Edges of the form u contribute $\frac{2^2+2^2}{49} = \frac{4}{49}$. In total, the coefficient is $\frac{42}{49}a^2$. Now consider the coefficient of ab. Edges of type p, q give coefficient $(-2\cdot3\cdot2\cdot2 - 2\cdot3\cdot1\cdot2 + 2\cdot1\cdot2\cdot2)/7^2 = -28/49$. Edges of type s, t give $-2\cdot2/49 = -4/49$. Edges of type r give $-2/49$. Edges of type u give $-2\cdot4/49 = -8/49$ The total is $-42/49$. Therefore by symmetry the total energy of the extension $\tilde{u}$ is

$$E_1(\tilde{u}) := E_{\Gamma_1}(\tilde{u}) = \frac{42}{49}(a^2 + b^2 + c^2 - ab - bc - ca) = \frac{3}{7}((a-b)^2 + (b-c)^2 + (c-a)^2).$$

This calculation has shown that the renormalisation equation holds with ratio

$$r = 3/7.$$

Henceforth, this is the constant that r will refer to in the context of the hexagasket.

This same process can be used to define a harmonic extension from V_m to V_{m+1} for each $m \geq 0$. On any m-cell w of any level, the algorithm to compute the harmonic extension of a function u' on the boundary of that cell to the vertices V_{m+1} inside that cell will be exactly the same as (3.4), (3.6) and so on. In other words, it is the problem of minimising $E_1(u')$ for extensions u' to V_1 of $u \circ F_w$. If we are given values of a function u on V_m, then any element of $V_{m+1} \backslash V_m$ belongs to a unique m-cell $F_w K$, so the total energy of an extension is the sum of the contributions from each m-cell $F_w K$:

$$E_{m+1}(u') = \sum_{|w|=m} E_1(u' \circ F_w).$$

So on each m-cell, the rules (3.4), (3.6) continue to hold, and the renormalisation factor stays as $r = 3/7$. Therefore we define

$$\mathcal{E}_m(u) = r^{-m} E_m(u)$$

(where $E_m := E_{\Gamma_m}$), so then the renormalised energy remains unchanged under harmonic extension ($\mathcal{E}_{m+1}(\tilde{u}) = \mathcal{E}_m(u)$), and cannot decrease for any extension, so

$$\mathcal{E}_0(u) \leq \mathcal{E}_1(u) \leq \mathcal{E}_2(u) \leq \cdots. \tag{3.7}$$

For an open, bounded, connected subset $U \subset \mathbb{R}^n$, a *harmonic* function $h : U \to \mathbb{R}$ can be defined, equivalently, as either those C^2 functions that minimise the Dirichlet energy $\int_U ||\nabla u||^2$ or as the C^2 solutions of Laplace's equation $\Delta u = 0$. In analogy with the energy minimising definition, we say that a function on V_* is *harmonic* if, given boundary values $u|_{V_0}$, u minimises $\mathcal{E}_m$ at all levels. In other words, the extension of u from V_m to V_{m+1} uses the rules (3.4) and (3.6), so $\mathcal{E}_0(u) = \mathcal{E}_1(u) = \mathcal{E}_2(u) = \cdots$. A function on K is said to be harmonic if it is continuous and its restriction to V_* is harmonic, noting that by the density of V_* in K there is at most one extension of a function on V_* to a continuous function on K. We will later define Δ_μ and show in Proposition 4.1 that harmonic functions on K are indeed precisely the solutions of $\Delta_\mu u = 0$ for any regular probability measure μ. It is clear that harmonic functions depend only on their values on the boundary V_0, just as in the classical case, suggesting that the name 'boundary' for V_0 is more sensible that it might at first seem. Harmonic functions are an important class of functions, and harmonic functions on the hexagasket will be explored further in Section 4.1.

The equality for harmonic extensions $\mathcal{E}_{m+1}(\tilde{u}) = \mathcal{E}_m(u)$ extends to the bilinear form by the polarisation identity (3.3). Indeed, if $\tilde{u}$ and $\tilde{v}$ are the harmonic extensions of u and v respectively, then by linearity of the minimising equations $\tilde{u} \pm \tilde{v}$ are the harmonic extensions of $u \pm v$, so

$$\begin{aligned}
\mathcal{E}_{m+1}(\tilde{u}, \tilde{v}) &:= r^{-(m+1)} E_{m+1}(\tilde{u}, \tilde{v}) \\
&= r^{-(m+1)} \frac{1}{4}(E_{m+1}(\tilde{u} + \tilde{v}) - E_{m+1}(\tilde{u} - \tilde{v})) \\
&= r^{-m} \frac{1}{4}(E_m(u + v) - E_m(u - v)) \\
&= r^{-m} E_m(u, v) \\
&=: \mathcal{E}_m(u, v)
\end{aligned}$$

where the first and last equalitites are definitions. In fact, Proposition 3.1 says that only one of the extensions must be harmonic for this to hold; the other can be arbitrary. Proposition 3.1 is from [45, p. 15]; there, Strichartz proves it for the Sierpinski gasket, but the proof for the hexagasket is almost identical.

Proposition 3.1. *If u, v are defined on V_m and $\tilde{u}$ is the harmonic extension of u and v' is an arbitrary extension of v to V_{m+1} then $\mathcal{E}_{m+1}(\tilde{u}, v) = \mathcal{E}_m(u, v)$.*

When there is an infinite increasing sequence like (3.7), the natural thing to do is to take the limit, as in [45, Section 1.4]. For any function u on V_*, $\mathcal{E}_m(u)$ will certainly converge to some limit $\mathcal{E}(u) \in [0, \infty]$ as $m \to \infty$. If u is constant then all the $\mathcal{E}_m$ will clearly equal 0 so $\mathcal{E}(u) = 0$, while if u is non-constant then (since V_* is dense in $K = H$) some $\mathcal{E}_m(u)$ will be nonzero and hence $\mathcal{E}(u) \neq 0$. The term to describe this property that nonconstant functions have positive energy that $\mathcal{E}$ is a *nondegenerate* energy.

Definition 3.1. The *domain of the energy* is the set E of all functions u on K with finite energy $\mathcal{E}(u) < \infty$, and E^0 is the subset of functions that vanish on the boundary.

We will now see that E is a dense subset of $C(K)$. The following proposition and proof is based on [45, p. 19].

Proposition 3.2. *Every $u \in E$ is uniformly continuous.*

Proof. For all $m \in \mathbb{N}$ we have that if x, y are adjacent vertices in Γ_m then one term in the sum defining $\mathcal{E}_m(u)$ is $r^{-m}(u(x) - u(y))^2 \leq \mathcal{E}_m(u) \leq \mathcal{E}(u) < \infty$. Therefore

$$|u(x) - u(y)| \leq r^{m/2}(\mathcal{E}(u))^{1/2}. \tag{3.8}$$

If $x_m, x_{m+1}, \ldots, x_{m+k}$ is a sequence of points such that $x_{m+j} \in V_{m+j}$ and $x_{m+j} \sim_{m+j+1} x_{m+j+1}$ for all j, then

$$|u(x_m) - u(x_{m+k})| \leq r^{m/2}(1 + r^{1/2} + \cdots r^{k/2})(\mathcal{E}(u))^{1/2} \leq \frac{r^{m/2}}{1 - r^{1/2}}(\mathcal{E}(u))^{1/2}.$$

by summing (3.8) along the sequence of edges using the triangle inequality. By the geometry of K, if $x, y \in V_*$ are in the same or adjacent m-cells, then they can be connected via two such chains, so

$$|u(x) - u(y)| = \frac{2r^{m/2}}{1 - r^{1/2}}(\mathcal{E}(u))^{1/2}, \tag{3.9}$$

giving the required uniform continuity. □

Note that the definition of energy only depends on $u|_{V_*}$, but that Proposition 3.2 and the density of V_* means that if the values of a function in E are specified on V_* then there will be a unique continuous extension to K. Proposition 3.2 also shows that $E \subset C(K)$. The inclusion is strict because (3.9) in fact gives a Hölder condition on u, so picking some $x \in K$ and $\alpha > 0$ that is smaller than the Hölder exponent, the function $u(y) = ||y - x||^\alpha$ will be continuous on K but cannot have finite energy.

Note that the polarisation identity for E_m extends to $\mathcal{E}_m$:

$$\mathcal{E}_m(u, v) = \frac{1}{4}(\mathcal{E}_m(u + v) - \mathcal{E}_m(u - v)).$$

If $u, v \in E$ then the right-hand side must have a limit, so the left-hand side will too, and so the polarisation identity holds for $\mathcal{E}$ as well. Now, assuming that all functions are continuous and defined on the whole of K, we have the following:

Theorem 3.3. *The quotient space $E/\langle$constant functions$\rangle$ is a Hilbert space with inner product*

$$\mathcal{E}(u, v) := \lim_{m \to \infty} \mathcal{E}_m(u, v).$$

Proof. We sketch the proof. The polarisation identity for $\mathcal{E}$ gives all the properties of an inner product, except positive definiteness. If $\mathcal{E}(u) = 0$ then for all $m \in \mathbb{N}$ we have $\mathcal{E}_m(u) = 0$, so u is constant on each V_m, so constant on V_*, so constant by the density of V_*. Therefore, factoring out the constants gives an inner product. All that is left to prove is that $E/\langle$constant functions$\rangle$ is complete with respect to the induced metric (as in [45, Theorem 1.4.2]). □

To be precise, we say $u_n \to u$ in energy as $n \to \infty$ if $\mathcal{E}(u_n - u) \to 0$ and $u_n \to u$ uniformly (as $n \to \infty$). To avoid the annoyance of having to factor out constants, we can define another inner product on E given by

$$\langle u, v\rangle := \langle u, v\rangle_{L^2(E,\text{Borel},\mu)} + \mathcal{E}(u, v).$$

Then if $\langle u, u\rangle = 0$ then u is 0 μ-almost everywhere, so if we identify functions that differ by a set of μ-measure 0, then the quotient space becomes a Hilbert space with inner product $\langle\cdot,\cdot\rangle$.

On the unit interval, piecewise-linear functions are useful as they can uniformly approximate any continuous function. The same will be true if we insist that the only points of non-differentiability of the piecewise-linear functions are at dyadic rationals, and we only need to specify the value of the function at the points of non-differentiability (including the endpoints) to determine the whole function. This leads us to an important analogous concept on PCF fractals:

Definition 3.2. A continuous function $u : K \to \mathbb{R}$ is a *piecewise harmonic spline of level m* if for all words w of length m, $u \circ F_w$ is harmonic.

The piecewise harmonic splines of level 0 are precisely the harmonic functions. Such a u is determined uniquely by specifying the values of u on V_m arbirarily and extending via the harmonic extension algorithm ((3.4) and (3.6) in the case of the hexagasket) to V_l for each $l > m$ (and then extending to K by continuity). Clearly for such a u we have that

$$\mathcal{E}_m(u) = \mathcal{E}_{m+1}(u) = \mathcal{E}_{m+2}(u) = \cdots$$

so $\mathcal{E}(u)$ must take the same value and is in particular finite. The next theorem ([45, Theorem 1.4.4] reworded) shows that E is dense in $C(K)$.

Theorem 3.4. *Let $u \in C(K)$ and for $m \in \mathbb{N}$ let u_m be the piecewise harmonic spline of level m defined by $u_m|_{V_m} = u|_{V_m}$. Then $u_m \to u$ uniformly as $m \to \infty$, and if $u \in E$ then $u_m \to u$ in energy.*

Now, from the definition of $\mathcal{E}_m$ we have that if $u \in E$ then

$$\mathcal{E}_{m+1}(u) = \sum_i r^{-1}\mathcal{E}_m(u \circ F_i).$$

Also, $u \in E$ means that $\mathcal{E}(u)$, the limit of the left-hand side, is finite, so the limit of each term on the right-hand side must also be finite, and the self-similar identity

$$\mathcal{E}(u) = \sum_i r^{-1}\mathcal{E}(u \circ F_i). \tag{3.10}$$

holds. By the polarisation identity we have that

$$\mathcal{E}(u, v) = r^{-1}\sum_i \mathcal{E}(u \circ F_i, v \circ F_i). \tag{3.11}$$

.

There is one more property of the energy worth noting.

Proposition 3.5 (Markov property)**.** *If $u \in E$ then $v := \min(\max(u, 0), 1) \in E$ and $\mathcal{E}(v, v) \leq \mathcal{E}(u, u)$.*

This follows from the corresponding property for $\mathcal{E}_m$, which follows from the corresponding property for E_m which follows because each term $(u(x) - u(y))^2$ in the sum (3.2) cannot increase.

A note of caution: generalising this method of renormalisation to other PCF fractals is not always possible. It has long been known that there are PCF SSS that don't have a self-similar energy for which (3.10) holds (below we will see that in fact more is true). Therefore it makes sense to ask which PCF fractals have an energy that satisfies a self-similar identity with different weights $r_i \in (0, 1)$:

$$\mathcal{E}(u) = \sum_i r_i^{-1} \mathcal{E}(u \circ F_i). \tag{3.12}$$

Such an energy is said to be *regular*, as is the harmonic structure resulting from such an energy. A way to construct such an energy would be by taking the limit of graph energies $\mathcal{E}_m$ defined on Γ_m of the form

$$\sum_{x \sim y} c(x, y)(u(x) - u(y))^2$$

for some conductances $c(x, y)$ on the edges of the graph. The conductances will have to be positive so that the resulting energy form will be nondegenerate, in other words so that constant functions will be the only functions with zero energy. The desired conductances will also be such that

$$\mathcal{E}_m(u) = \sum_i r_i^{-1} \mathcal{E}_m(u \circ F_i)$$

for all $m \geq 0$ so that the self-similar identity (3.12) holds in the limit. The conductances must also be such that for any function u defined on V_{m-1}, the extension $\tilde{u}$ of u to V_m that minimises $\mathcal{E}_m$ among all extensions of u to V_m will satisfy $\mathcal{E}_m(\tilde{u}) = \mathcal{E}_{m-1}(u)$. In [45, Section 4.2], Strichartz reformulates this problem as a nonlinear Perron-Frobenius eigenvalue problem. In 2019, Peirone [38] constructed a PCF SSS on which there is no choice of r_i for which there exists a self-similar energy, solving what had been a significant open problem in analysis on fractals. Those fractals for which there exist r_i that give a self-similar energy have not yet been fully characterised, but there are partial results; in [31] Lindstrøm proves that there exists such an energy on all *nested* fractals (defined in Definition 2.18) with all weights equal to 1.

From the above analysis it is clear that the energy that has been constructed on H is an example of a more general class of functionals, studied in detail in [17], called Dirichlet forms.

Definition 3.3. A *Dirichlet form* on a measure space (X, Σ, μ) is a positive semidefinite bilinear form $\mathcal{E} : E \times E \to \mathbb{R}$ defined on a dense subset $E \subseteq L^2(X, \Sigma, \mu)$ that satisfies the following properties:

(Hilbert space) E is a Hilbert space with the inner product $\langle u, v \rangle = \langle u, v \rangle_{L^2(X,\Sigma,\mu)} + \mathcal{E}(u, u)$.

(Markov property) If $u \in E$ then $v := \min(\max(u, 0), 1) \in E$ and $\mathcal{E}(v, v) \leq \mathcal{E}(u, u)$.

Note that E is a dense subset of $C(K)$ which is a dense subset of $L^2(K, \text{Borel}, \mu)$. A Dirichlet form $\mathcal{E}$ is called *nondegenerate* if $\mathcal{E}(u) = 0$ implies that u is constant almost everywhere. The corresponding quadratic form can also be called a Dirichlet form, just as the quadratic form corresponding to the energy we constructed can also be called an energy.

3.3 Weak formulation of the Laplacian

If we have a regular probability measure μ and an energy $\mathcal{E}$ on a SSS K, then we can define a weak formulation of the Laplacian in analogy with the classical case (3.1). Recall that E^0 is as in Definition 3.1.

Definition 3.4. Let the *domain of the Laplacian* $L = L_\mu \subseteq E$ be the set of functions $u \in E$ such that there is a continuous $f : K \to \mathbb{R}$ such that for all $v \in E^0$ we have

$$\mathcal{E}(u, v) = -\int_K f v d\mu.$$

In this case, we write $\Delta_\mu u = f$ and say that the (weak) Laplacian of u with respect to the measure μ is f. If merely $f \in L^2(K, \text{Borel}, \mu)$, then $u \in L_{L^2}$.

This definition works equally well when $\mathcal{E}$ is any Dirichlet form on E. It is highly dependent on the measure μ; the positivity condition on μ ensures that μ 'sees' all of K. If f_1 and f_2 are distinct continuous functions then there is a ball B in K on which $|f_1 - f_2|$ is strictly positive, so if v is a non-negative, piecewise-harmonic spline with support inside this ball then $\int_K f_1 v d\mu \neq \int_K f_2 v d\mu$. If we assume that B is small enough then $v \in E^0$ so f_1 and f_2 cannot both be the Laplacian of u, so Definition 3.4 determines the Laplacian uniquely.

It follows immediately from bilinearity of $\mathcal{E}$ and linearity of the integral that if $c \in \mathbb{R}$ and $u, v \in L$ then $cu + v \in L$ and $\Delta_\mu(cu + v) = c\Delta_\mu u + \Delta_\mu v$. This shows both that L is a real vector space and that Δ_μ is a linear map $L \to C(K)$. Every constant function u is certainly in L with $\Delta_\mu u = 0$, but this is all that is immediately obvious about L. We wait until Chapter 4 to show that L is large enough to be interesting; first we prove some properties about functions $u \in L$ without actually finding any such functions. Assume now that μ is self-similar, as in (2.4). We also assume that the self-similar equation (3.11) for the energy holds. Then, as in [45, p. 33], by self-similarity and Definition 3.4 we have that if $u \in L, \Delta u = f, v \in E^0$ then

$$-\sum_i r^{-1}\mathcal{E}(u \circ F_i, v \circ F_i) = \sum_i \mu_i \int_K (f \circ F_i)(v \circ F_i) d\mu.$$

Therefore for each i, if $y \in E^0$ then we can apply this equation with $v \in E^0$ defined such that $v \circ F_j = y$ and $v \circ F_i = 0$ for all $j \neq i$, to get

$$-r^{-1}\mathcal{E}(u \circ F_j, y) = \mu_j \int_K (f \circ F_j) y d\mu.$$

Thus by definition $u \circ F_i \in L = L_\mu$ with

$$\Delta(u \circ F_j) = r\mu_j(\Delta u) \circ F_j$$

(where $\Delta = \Delta_\mu$). For any word w we can set $r_w := r^{|w|}$ and iterate to give the scaling property

$$\Delta(u \circ F_w) = r_w \mu_w (\Delta u) \circ F_w.$$

When μ is the standard measure then for the hexagasket H, the scaling factor is $r\mu_j = \frac{3}{7} \times \frac{1}{6} = \frac{1}{14}$. The classical Laplacian on $\mathbb{R}^n$ is a second-order differential operator; the chain rule gives

$$\Delta\left(u\left(\frac{1}{3}x\right)\right) = \left(\frac{1}{3}\right)^2 \Delta(u)\left(\frac{1}{3}x\right) = \frac{1}{9}\Delta(u)\left(\frac{1}{3}x\right).$$

This doesn't depend on n, so the particular value $\frac{1}{14}$ should not be a reflection of the dimension of the hexagasket. Certainly $\frac{1}{14} \neq \frac{1}{9}$, so the conclusion is that it would be wrong to think of the Laplacian that we have constructed on H as a *second order* operator. In [45, p. 85], Strichartz argues that there is a way of defining the order of the Laplacian using a natural intrinsic metric called the *effective resistance metric* and the *spectral dimension* of the set (which will be defined in Section 5.1), such that the spectral asymptotics are in some sense compatible with smooth analysis.

3.4 Pointwise formula

We now give a pointwise formula for the Laplacian that we have developed, as in [45, Section 2.2], based on the second difference quotient formula for the second derivative on $\mathbb{R}$ (which is, of course, the Laplacian on $\mathbb{R}$):

$$\frac{u\left(x + \frac{1}{2^m}\right) - 2u(x) + u\left(x - \frac{1}{2^m}\right)}{\left(\frac{1}{2^m}\right)^2}.$$

Such a formula will show that the Laplacian at a point depends only on the values of the function in any neighbourhood of that point. We do this by approximating Δ_μ with renormalised discrete 'Laplacians' defined on the graphs Γ_m. We define such a graph Laplacian by

$$\Delta_m u(x) = \sum_{y \sim_m x} (u(y) - u(x)) \tag{3.13}$$

for each $x \in V_m \backslash V_0$. This measures the average difference in value of u at the point x from the neighbours of x.

The way to derive the correct formula for points $x \in V_m \backslash V_0$ from the weak formulation is to substitute $v = \psi_x^{(m)}$ into Definition 3.4, where $\psi_x^{(m)}$ is the piecewise-harmonic spline of level m defined by $\psi_x^{(m)}(y) = \delta_{xy}$ for $y \in V_m$ (for convenience, we use the convention that if $x \in K \backslash V_m$ then $\psi_x^{(m)}$ is identically 0). Note that $\psi_x^{(m)}$ is constant (zero) on all edges that do not contain x, so

$$-r^{-m}\Delta_m u(x) = \mathcal{E}_m(u, \psi_x^{(m)}) = \int_K f\psi_x^{(m)} d\mu \approx f(x) \int_K \psi_x^{(m)} d\mu, \tag{3.14}$$

where the approximate equality holds because the support of $\psi_x^{(m)}$ is close to x and f is continuous. This suggests that the right formula is the one in the next theorem, where we will make this argument precise and prove that the weak and pointwise definitions of the Laplacian are equivalent. The next theorem and proof are similar to [45, Theorem 2.2.1] for the Sierpinski gasket, but we add in the two omitted arguments related to uniform convergence.

Theorem 3.6 (Pointwise Laplacian)**.** *If $u \in L$ then*

$$f_m(x) := r^{-m}\left(\int_K \psi_x^{(m)} d\mu\right)^{-1} \Delta_m u \to \Delta_\mu u(x) \tag{3.15}$$

uniformly for $x \in V_*\backslash V_0$, as $m \to \infty$. Conversely, if u is a continuous function such that the right-hand side of (3.15) *holds uniformly for $x \in V_*\backslash V_0$ then $u \in L$ and* (3.15) *holds.*

Proof. If $u \in L$ and $\Delta u = f$ then (3.14) gives that

$$r^{-m}\left(\int_K \psi_x^{(m)} d\mu\right)^{-1} \Delta_m u = \frac{\int_K f\psi_x^{(m)} d\mu}{\int_K \psi_x^{(m)} d\mu}. \tag{3.16}$$

The function f is continuous on the compact set K, hence uniformly continuous, so given $\epsilon > 0$ there exists $\delta > 0$ such that if $||y - x|| < \delta$ then $|f(y) - f(x)| < \epsilon$. There is an $M \in \mathbb{N}$ such that if $m > M$ then for all $x \in V_*\backslash V_0$, each element y of the support of $\psi_x^{(m)}$ satisfies $||y - x|| \le \delta$, and so

$$f(x) - \epsilon \le \frac{\int_K f\psi_x^{(m)} d\mu}{\int_K \psi_x^{(m)} d\mu} \le f(x) + \epsilon,$$

proving the uniform convergence of the right-hand side of (3.16) to f.

Conversely, suppose that u is a continuous function such that the right-hand side of (3.15) converges uniformly to some continuous function f. Then if $v \in E^0$ we have

$$\begin{aligned}
\mathcal{E}_m(u,v) &= r^{-m} \sum_{x\sim_m y} (u(y) - u(x))(v(y) - v(x)) \\
&= -r^{-m} \sum_{x\in V_m\backslash V_0} v(x) \left(\sum_{x\sim_m y} (u(y) - u(x))\right) \\
&= -\sum_{x\in V_m\backslash V_0} v(x) r^{-m} \Delta_m u(x) \\
&= -\sum_{x\in V_*\backslash V_0} v(x) \left(\int_K \psi_x^{(m)} d\mu\right) f_m(x) \\
&= -\int_K \left(\sum_{x\in V_*\backslash V_0} v(x) f_m(x) \psi_x^{(m)}\right) d\mu,
\end{aligned}$$

where we have collected all the terms in the sum that contain the factor $v(x)$ and noted that each edge appears only once in the sum in the first line. The conclusion follows upon taking the limit $m \to \infty$ if we can show that $\sum_{x\in V_*\backslash V_0} v(x) f_m(x) \psi_x^{(m)}(y)$ converges to $v(y)f(y)$ uniformly for $y \in K$ as $m \to \infty$; we now show that this is indeed the case. The function v is continuous on a compact set, so bounded, say $||v||_\infty \le M < \infty$. As $f_m \to f$ uniformly by assumption, given $\epsilon > 0$ there is $M \in \mathbb{N}$ such that if $m \ge M$ then $||f_m - f||_\infty < \epsilon/M$. So if $y \in V_*\backslash V_0$ and $m \ge M$ then since only the term corresponding to $x = y$ in the sum remains, we have

$$\left|\sum_{x\in V_*\backslash V_0} v(x) f_m(x) \psi_x^{(m)}(y) - v(y)f(y)\right| = |v(y)f_m(y) - v(y)f(y)| \le ||v||_\infty ||f_m - f||_\infty < \epsilon,$$

giving the required uniform convergence on $V_*\backslash V_0$. To use this to show uniform convergence on K, note that the product vf is continuous hence uniformly continuous, so there is $\delta > 0$ such that if

$||y-x|| < \delta$ then $|v(y)f(y) - v(x)f(x)| < \epsilon$. There is $M' \geq M$ such that if $m \geq M'$ then every m-cell has diameter less than δ. If $y \in K\backslash(V_*\backslash V_0)$ and C is any m-cell containing y with $m > M'$, then for any boundary vertex x of C, we have that

$$|v(y)f(y) - v(x)f_m(x)| \leq |v(y)f(y) - v(x)f(x)| + |v(x)f(x) - v(x)f_m(x)| < 2\epsilon.$$

Thus

$$\left| \sum_{x \in V_*\backslash V_0} v(x)f_m(x)\psi_x^{(m)}(y) - v(y)f(y) \right| \leq 6\epsilon$$

since the sum extends over only those vertices x that form the boundary of C, and $\sum_{x\in V_*\backslash V_0} \psi_x^{(m)}(y)$ is identically 1 on C. This gives the required uniform convergence, completing the proof. □

As noted in [45, p. 36], by the uniform continuity, it is clear that for $x \in K\backslash(V_*\backslash V_0)$, if (x_m) is any sequence converging to x with $x_m \in V_m\backslash V_0$, then the pointwise formula

$$r^{-m}\left(\int_K \psi_{x_m}^{(m)} d\mu\right)^{-1} \Delta_m u(x_m) \to \Delta_\mu u(x)$$

as $m \to \infty$ holds.

When μ is the standard measure, the value of $\int_K \psi_x^{(m)} d\mu$ for $m \geq 1$ is computed for the Sierpinski gasket; we follow a similar method to compute it for the hexagasket H with the standard measure μ, though it is slightly more complicated because on the Sierpinski gasket all elements of $V_m\backslash V_0$ have the same valency (which is 4). On the hexagasket, it is straightforward to see by induction that vertices in Γ_m have valency either 2 (if it is a boundary point or a non-junction point and so lies in only one m-cell) or 4 (if it is a junction point and lies in two m-cells). Moreover, if $x \in V_m$ then the valency of x is the same in each of the graphs $\Gamma_m, \Gamma_{m+1}, \Gamma_{m+2}, \ldots$ so we may write $\mu_0(x)$ for this valency, and this is independent of m. For convenience let $m > 1$, for then there is an m-cell F_wK bounded by vertices x, y, z such that each of x, y and z has valency 4. Now, $\psi_x^{(m)} + \psi_y^{(m)} + \psi_z^{(m)}$ is harmonic and on F_wK and is 1 at the three vertices, so is identically 1 on F_wK. By symmetry, $\int_K \psi_x^{(m)} d\mu = \int_K \psi_y^{(m)} d\mu = \int_K \psi_z^{(m)} d\mu$, so

$$3\int_{F_wK} \psi_x^{(m)} d\mu = \int_{F_wK} (\psi_x^{(m)} + \psi_y^{(m)} + \psi_z^{(m)}) d\mu = \mu(F_wK) = \frac{1}{6^m}.$$

Thus $\int_{F_wK} \psi_x^{(m)} d\mu = \frac{1}{3}\frac{1}{6^m}$, so with the contribution of the other m-cell we have $\int_K \psi_x^{(m)} d\mu = \frac{2}{3}\frac{1}{6^m}$, and so

$$\left(\int_K \psi_x^{(m)} d\mu\right)^{-1} = \frac{3}{2}6^m = \frac{3}{2}3^m 2^m.$$

If v is a vertex with valency 2 and x is as above then clearly

$$\int_K \psi_v^{(m)} d\mu = \frac{1}{2}\int_K \psi_x^{(m)} d\mu = \frac{1}{3}\frac{1}{6^m},$$

so

$$\left(\int_K \psi_v^{(m)} d\mu\right)^{-1} = 3 \cdot 3^m 2^m.$$

Also, $r^{-m} = 7^m/3^m$ so

$$6 \cdot 14^m \tilde{\Delta}_m u(x) = \frac{6}{\mu_0(x)} 14^m \Delta_m u(x) \to \Delta u(x), \tag{3.17}$$

where $\tilde{\Delta}_m(x) := \frac{1}{\mu_0(x)} \sum_{y \sim_m x} (u(y) - u(x))$ is the weighted graph Laplacian. We can imagine we are assigning a weight $\mu_0(x)$ to each vertex x, so that Δ is a renormalised limit of $\tilde{\Delta}_m$ with no extra weights. The constant 6 clearly makes no difference to the theory, but as Strichartz [45, p. 36] notes in the case of the Sierpinski gasket, the value 14 is very important. Replacing it with a smaller value would result in the Laplacian of most functions in L being 0, while replacing it with a larger constant will result in the limit (3.17) not existing for most functions in L.

3.5 Random walks

We now look at the probabilistic approach to defining a Laplacian. We discuss a random walk on the hexagasket H in a similar way to how Falconer discusses a random walk on the Sierpinski gasket in [11, Section 12.4]. A key idea is Brownian motion on $\mathbb{R}^n$, named after Robert Brown, a botanist who observed the motion of small dust particles. It has been studied in detail, for example by Einstein [8] as early as 1905, who realised that the apparently random motion was the result of many collisions between the dust particles and fast-moving water molecules; this is one physical process that Brownian motion is effective at modelling. It is closely related to diffusion processes, some physical applications of which are listed in the introduction to Chapter 5. Brownian motion can be defined by considering the cubic lattice $S_k = \{(x_1 2^{-k}, \dots, x_n 2^{-k}) | x_1, \dots, x_n \in \mathbb{Z}\}$. Consider the random walk on this set starting at $X(0) = 0$ with unit time steps such that at each time $t \in \mathbb{Z}$ we increase or decrease one of the x_i by $\pm 2^{-k}$, moving to one of the $2n$ possible places with equal probability. This is a 'memoryless' random walk (a Markov chain). It may be shown from the geometry of the set that the expected distance from the starting point after time m is of the order $O(\sqrt{m})$. Thus if we let X_t be the random walk as above but where now steps are taken at time intervals 4^{-k} (which is crucially the square root of the step size 2^{-k}), then it can be shown that $X_n(t)$ converges as $k \to \infty$ to a continuous-time stochastic process on $\mathbb{R}^n$, namely *Brownian motion*. The memoryless property still holds, so for all $t, h \geq 0$ the increments $X(t+h) - X(t)$ are independent. Throughout, we will omit the (often technical and involved) details of how the approximate random processes we construct on graphs converge to a random process on the object of interest itself, and the proofs of the properties of these limit processes.

To construct an analogous process on H we define a simple random walk $X_m(t)$ on Γ_m to be the Markov chain that takes steps in some time intervals α_m with equal probability, so that the transition probabilities are given by

$$\mathbb{P}^{\cdot}(X((k+1)\alpha_m) = y | X(k\alpha_m) = x) = \frac{1}{\mu_0(x)}$$

if $y \sim_m x$, and 0 otherwise, where $\mathbb{P}^x$ is the probability function conditional on $X_0 = x$. If $m \geq 1$, then X_m induces a random walk on X_{m-1} by noting the sequence of vertices of Γ_{m-1} visited, ignoring consecutive occurrences of the same vertex. By the symmetry of the graphs, this will move to any of

the adjacent vertices in Γ_{m-1} with equal probability, so this is just $X_{m-1}(t)$ with steps but where the time interval between each step now varies. Therefore for this to converge to a limiting process on H, we want this induced walk to be close to X_{m-1} at large scales, so it makes sense to choose the time intervals α_m so that the *expected* time for Γ_m to move to a neighbouring vertex in Γ_{m-1} is α_{m-1}. Note that by symmetry, the expected time will not depend on which vertex $x \in \Gamma_{m-1}$ we start at (it will be the same whether or not x is a junction point). Without loss of generality, let $m = 1$, let x be q_1, and use the notation in Figure 3, so that we are looking for the expected number of steps that $X_1(t)$ takes to reach one of the other boundary vertices q_3 or q_5. For a vertex $v \in \Gamma_m = \Gamma_1$, write $E(v)$ for the expected number of steps that $X_m(t)$ takes to reach q_3 or q_5 starting at v (so we seek to find $E(q_1)$). By symmetry $E(z) = E(z')$ and $E(x) = E(x')$ so we may assume that the first step taken by X_m is to z (rather than z'). Thus

$$E(q_1) = 1 + E(z).$$

From z, X_m can move to any of the four neighbours of z with equal probability so, again using $E(z) = E(z')$, we have

$$E(z) = 1 + \frac{1}{4}E(q_1) + \frac{1}{4}E(z) + \frac{1}{4}E(q_2) + \frac{1}{4}E(y).$$

Continuing this way and repeatedly exploiting symmetry, we have

$$\begin{aligned}
E(q_2) &= 1 + \frac{1}{2}E(z) + \frac{1}{2}E(y) \\
E(y) &= 1 + \frac{1}{4}E(z) + \frac{1}{4}E(q_2) + \frac{1}{4}E(x) + \frac{1}{4} \times 0 \\
E(x) &= 1 + \frac{1}{4}E(y) + \frac{1}{4} \times 0 + \frac{1}{4}E(x) + \frac{1}{4}E(q_4) \\
E(q_4) &= 1 + E(x).
\end{aligned}$$

Simplifying the above equations and solving the system of linear equations by Gaussian elimination yields the solution

$$E(q_1) = 14; \quad E(z) = 13; \quad E(q_2) = 12; \quad E(y) = 9 \quad E(x) = 7; \quad E(q_4) = 8.$$

These answers can quickly be verified by substituting them back into the linear equations, and their relatives magnitudes are as would be expected from the geometry of the problem. Thus the mean time interval of X_m on X_{m-1} is $14\alpha_m$, so as discussed above, we want to try taking $\alpha_{m-1} = 14\alpha_m$. Then (up to a constant factor which will not be important) $\alpha_m = 14^{-m}$. It can then be shown, as with standard Brownian motion on $\mathbb{R}^n$, that X_t converges to a random process $X(t)$ on H, called *Brownian motion* on H. This is discussed in great detail for the example of the Sierpinski gasket in [5].

Now, (3.17) means that

$$6\frac{\tilde{\Delta}_m u(x)}{\alpha_m} \to \Delta u(x) \tag{3.18}$$

as $m \to \infty$ for any function $u \in L$, and we will see in Section 5.2 that the same number 14 is needed in the renormalisation of both the pointwise formula for the Laplacian and for the random walks. Let $\mathbb{E}^x$ be the expectation with respect to $\mathbb{P}^x$. For any function u on Γ_m and any $x \in \Gamma_,$, we have that

$$\mathbb{E}^x[u(X_m(\alpha_m)) - u(X_m(0))] = \frac{1}{\mu_0(x)} \sum_{y \sim_m x} (u(y) - u(x)) = \tilde{\Delta}_m u(x).$$

Thus (3.18) says that

$$6\frac{\mathbb{E}^x[u(X_m(\alpha_m)) - u(x)]}{\alpha_m} \to \Delta u(x)$$

as $m \to \infty$ for any $u \in L$. This suggests that

$$6\frac{\mathbb{E}^x[u(X(t))] - u(x)}{t} = 6\frac{\mathbb{E}^x[u(X(t)) - u(x)]}{t} \to \Delta u(x)$$

as $t \to 0^+$. On any set where there is a stochastic process X analogous to Brownian motion, we can *define* a Laplacian (up to an unimportant constant) to be the *infinitesimal generator of* X, which is the operator Δ given by

$$\Delta u(x) := \lim_{t \to 0^+} \frac{\mathbb{E}^x[u(X(t))] - u(x)}{t} \tag{3.19}$$

for all u such that the limit exists in a suitable sense, either pointwise or in a suitable norm. We will see in Section 5.2 that this can be done on the unbounded hexagasket, for example. This is the essence of how, historically, Laplacians were first defined on fractal domains. For example, [6] discusses the construction of Brownian motion on Sierpinski carpets, which are not PCF. This is the type of definition of the Laplacian which we will use when discussing the heat equation on unbounded fractal domains in Chapters 5 and 6.

4 Laplace's and Poisson's equations

Now that a Laplacian has been constructed, it is natural to consider two linear equations, namely Laplace's equation and the more general Poisson's equation. We carry out the relevant analysis and computations for the hexagasket; some of the methods are similar to those for the Sierpinski gasket in [45]. This section is necessary to establish something fundamental about the Laplacian: we have seen that L is a vector space that contains the constant functions, but, as Strichartz notes for the Sierpinski gasket in the introduction of [45, p. x], this is about all that can be said about L without further analysis. If L were one-dimensional, or even finite dimensional, then the whole of the theory of the previous chapter would be rather boring and pointless. When studying Laplace's equation, we will show that the subspace of L that are the solutions of Laplace's equation has dimension equal to the (finite) cardinality of the boundary of the fractal. When studying Poisson's equation, we will show that in fact L is infinite dimensional.

4.1 Laplace's equation and harmonic structure

As promised, we now show that for any regular probability measure μ, harmonic functions are the same as solutions of Laplace's equation $\Delta_\mu u = 0$, just as in the classical case. We say that $u : K \to \mathbb{R}$

is a solution of Laplace's equation if $u \in L$ and $\Delta_\mu u = 0$. The following theorem and proof are based on [45, Theorem 2.1.2] for the Sierpinski gasket.

Theorem 4.1. *Let μ be any regular probability measure on the hexagasket K. A function $u : K \to \mathbb{R}$ is harmonic if and only if it is a solution of Laplace's equation.*

Proof. Let u be harmonic. If $v \in E^0$ then $\mathcal{E}_0(u, v) = 0$. Proposition 3.1 then tells us that $0 = \mathcal{E}_0(u, v) = \mathcal{E}_1(u, v) = \mathcal{E}_2(u, v) = \cdots$ so $\mathcal{E}(u, v) = 0$, proving that u is a solution of Laplace's equation.

Conversely, suppose u solves Laplace's equation, and let $\psi_x^{(m)}$ be as in (3.14) for some $x \in V_m \backslash V_0$ and $m \in \mathbb{N}$. Then $\psi_x^{(m)} \in E^0$ and $\Delta_\mu u = 0$ so $\mathcal{E}(u, \psi_x^{(m)}) = 0$ by the definition of the weak Laplacian. Thus by Proposition 3.1 we have that $0 = \mathcal{E}(u, \psi_x^{(m)}) = \mathcal{E}_m(u, \psi_x^{(m)})$ so $\sum_{y \sim_m x}(u(x) - u(y)) = 0$ and so $u|_{V_m}$ is harmonic. As m was arbitrary, u is harmonic, as required. □

We have looked at harmonic functions already in Section 3.2 because the definition of a harmonic function is so closely linked with the definition of energy. Now that we have proven that harmonic functions are precisely the solutions of Laplace's equation, we analyse them more closely. We have seen that on the PCF fractals K that we have considered, like the hexagasket, given any function on $V_0 \to \mathbb{R}$, there is unique harmonic extension to each Γ_m, so there is a unique harmonic function on V_* with these boundary values. Harmonic functions are certainly elements of E, so by Proposition 3.2 they are uniformly continuous on V_*, and so have a unique continuous extension to K. Thus harmonic functions on K are also determined uniquely by their boundary values. By the linearity of the minimising equations (which are the partial derivatives of the quadratic function $E_m(u)$), if $\tilde{u}$ and $\tilde{v}$ are the harmonic extensions of u and v at some level m, and $c \in \mathbb{R}$, then $c\tilde{u} + \tilde{v}$ is the harmonic extension of $cu + v$. Thus, the set $\mathcal{H}$ of harmonic functions forms a $|V_0|$-dimensional vector space, which by Theorem 4.1 is a subspace of L, showing that L at least contains more than just the constant functions.

A natural basis for $\mathcal{H}$ is $\{h_0, \ldots, h_{|V_0|}\}$ defined by $h_j(q_i) = \delta_{ij}$ for $q_j \in V_0$. The harmonic extension algorithm can be described using matrices $A_0, \ldots, A_{|V_0|-1}$, called *harmonic extension matrices*, describing what happens to the boundary values as we move from a cell of level m to its sub-cells of level m_1. As in [45, p. 16], regarding a set of h-values as a vector of length $|V_0|$, we want A_i to be the matrix representing the linear map $h \mapsto h \circ F_i$ with respect to the basis $\{h_0, \ldots, h_{V_0}\}$, so

$$h|_{F_i V_0} = A_i h|_{V_0} \qquad \text{and} \qquad h|_{F_w F_i V_0} = A_i h|_{F_w V_0}$$

for any word w, the latter simply being the former applied to the harmonic function $u \circ F_w$ (the 'blowup' of u corresponding to the word w). For the hexagasket, let h_1 correspond to F_1 (vertex a on Figure 3), h_2 to F_3 (vertex b), and h_3 to F_5 (vertex c). The extension algorithm (3.4), (3.6) that we have already computed shows that there are three matrices corresponding to the 1-cells

K_1, K_3, K_5 that contain boundary vertices, that are cyclic permutations of each other:

$$A_1 = \frac{1}{7}\begin{pmatrix} 7 & 0 & 0 \\ 4 & 2 & 1 \\ 4 & 1 & 2 \end{pmatrix}; \qquad A_3 = \frac{1}{7}\begin{pmatrix} 0 & 7 & 0 \\ 1 & 4 & 2 \\ 2 & 4 & 1 \end{pmatrix}; \qquad A_5 = \frac{1}{7}\begin{pmatrix} 0 & 0 & 7 \\ 2 & 1 & 4 \\ 1 & 2 & 4 \end{pmatrix}.$$

There are three more matrices corresponding to the 1-cells K_2, K_4, K_6 that do *not* contain boundary vertices, and these are also cyclic permutations of each other:

$$A_2 = \frac{1}{7}\begin{pmatrix} 3 & 3 & 1 \\ 2 & 4 & 1 \\ 4 & 2 & 1 \end{pmatrix} \qquad A_4 = \frac{1}{7}\begin{pmatrix} 1 & 3 & 3 \\ 1 & 2 & 4 \\ 1 & 4 & 2 \end{pmatrix} \qquad A_6 = \frac{1}{7}\begin{pmatrix} 3 & 1 & 3 \\ 4 & 1 & 2 \\ 2 & 1 & 4 \end{pmatrix}. \tag{4.1}$$

Now, the linear combination of the columns

$$\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} - 6\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

so A_2 is non-invertible, and similarly for A_4 and A_6. We say that a PCF SSS has a *degenerate harmonic structure* if at least one of the matrices A_i is singular, so using this terminology we have shown that the hexagasket H has a degenerate harmonic structure. Any pair of rows are clearly linearly independent, so each of these three matrices has rank 2. This means that the space of restrictions of harmonic functions to F_4K is 2-dimensional (and similarly for F_2K and F_6K). We already noted that it was not three dimensional when we used a symmetry argument to show that a non-constant harmonic function was constant on F_4K. The value of a harmonic function on the boundary of the cell K_wK is clearly given by $h|_{F_wV_0} = A_w h|_{V_0}$ where $A_w = A_{w_{|w|}} \cdots A_{w_1}$. For the Sierpinski gasket, all the matrices A_1, A_2, A_3 are invertible, so on any cell the space of harmonic functions will be three-dimensional. Note that on any PCF SSS the space will be at least 1-dimensional, since it contains the subspace of constant functions, as every constant function on a cell is trivially the restriction of the same constant function on the whole set. An interesting question would be to work out on which cells of the hexagasket the space of restrictions of harmonic functions to that cell is 3-D, on which cells it is 2-D, and on which cells, if any (a big 'if'), it is 1-D. If the word corresponding to a cell contains at least one '2', '4' or '6', then the space cannot be 3-D, so in some sense the dimension is less than three on 'most' cells. One can ask whether the particular 2-D subspace can determine something about the location of the cell. The same question of what the subspace can say about the location of the cell can be asked of any PCF SSS with a degenerate harmonic structure. It is straightforward to determine the space of restrictions of harmonic functions to any *given* cell (just multiply out the matrices corresponding to the word of the cell), but *characterising* all the cells where the space has a particular dimension (or equivalently characterising which finite sequences of the A_i have a certain rank) is difficult. This is an example of a concept that Strichartz calls 'geography is destiny': the *location* on a fractal is related to analytical properties like the local structure of harmonic functions at that location. There is certainly no

analogue of this principle on locally homogeneous domains like open, connected subsets of $\mathbb{R}^n$. There is also a probabilistic version of the geography is destiny principle that holds even when the harmonic structure is not degenerate: if we choose a long product (corresponding to a word w of length m, say) of the matrices A_i with equal probabilities, then in many cases it can be shown using the theory of random matrices that for 'most' words w the image of the matrix product will be in some sense 'close' to some proper subspace. This means that for 'most' cells of large word length, the restriction of a harmonic function to that cell is 'likely' to lie 'close' to a certain subspace. Öberg, Strichartz and Yingst explore both of these forms of the geography is destiny principle for the hexagasket in [37], but under one crucial assumption that is equivalent to the assertion that the space of harmonic functions on the hexagasket is never one-dimensional (i.e. products of the $A_i, \dots A_6$ never have rank 1). If the assertion were false then that would mean that there are cells on the hexagasket where *all* harmonic functions restrict to a constant function, which would be very surprising indeed. This does not seem to have been proven, and from personal experience it seems quite difficult to prove. The action of the matrices on the 2-dimensional quotient space in which functions that differ by a constant are deemed equivalent can be considered, giving 2×2 matrices. The problem then becomes whether a product can move the 1-dimensional image of one of the non-invertible 2×2 matrices into the kernel of another - this is asking about the action of the semigroup generated by the six matrices on projective 1-space.

We now briefly discuss what the spectrum of the harmonic extension matrices says about harmonic functions on the hexagasket, as was done in [45, p. 17] for the Sierpinski gasket. In each matrix, each row sums to 1, so $\begin{pmatrix}1 & 1 & 1\end{pmatrix}^T$ is an eigenvector (corresponding to the constant functions) with corresponding eigenvalue 1. A standard calculation gives the other eigenvectors of A_1 as $\begin{pmatrix}0 & 1 & 1\end{pmatrix}^T$ corresponding to the function $h_3 + h_5$, with eigenvalue $3/7 = r$, and $\begin{pmatrix}0 & 1 & -1\end{pmatrix}^T$ corresponding to the function $h_3 - h_5$, with eigenvalue $1/7$. The differing eigenvalues mean that these two functions decay at different rates; on the m-cell $F_1^m K$, we have that $|h_3 + h_5| = O((3/7)^m)$ while $|h_3 - h_5| = O(1/7^m)$. Note that the former eigenvector is symmetric with respect to reflection that fixes q_1 and interchanges q_3 and q_5, while the latter is antisymmetric. These two vectors form a basis for the 2-dimensional subspace of harmonic functions that take the value 0 at q_0. 'Most' harmonic functions that are 0 at q_1 will have a symmetric component, and so decay with order $O((3/7)^m)$.

There is a physical interpretation of harmonic functions, namely the electric network interpretation, which is explored in [45, Chapter 1.5]. Suppose we have a finite set of nodes, with some pairs joined by an Ohmic resistor of resistance 1 Ohm, forming a graph G. Suppose we attach batteries of appropriate strength to each node so that each node x stays at voltage $u(x)$ (relative to the ground, say). By Ohm's law, a current of $(u(x) - u(y))$ Ampere will flow through the resistor between nodes x and y, so since potential difference is the work done per unit charge, an energy of $(u(x) - u(y))^2$ will be dissipated by the resistor between x and y each second. This means that the total energy dissipated each second will be as in (3.2), indicating that 'energy' is an appropriate word for the quantity defined in (3.2). If we instead only attach batteries to impose voltages at certain vertices, then by the physical laws of electrical network theory, the voltages at the other vertices will settle

into values that minimise the rate of energy dissipation. Thus, if the graph is Γ_m for some $m \in \mathbb{N}$ for some PCF SSS, and the voltage is imposed only at the boundary vertices V_0, then the voltages at the other vertices will be just as if the harmonic extension algorithm had been applied. In [45, Chapters 1.5 and 1.6], Strichartz considers more general networks where the resistances are not all 1, showing that the principles of electric network theory give an alternative way of *deriving* the harmonic extension algorithm for the Sierpinski gasket, and uses the electric network interpretation to define a natural intrinsic metric (the 'effective resistance' metric) on the Sierpinski gasket.

4.2 Poisson's equation

We now turn to Poisson's equation $\Delta u = f$, where the nonzero forcing term f makes the PDE inhomogeneous. We first give a rough idea of how to solve this equation on an open, connected domain $U \subset \mathbb{R}^n$ with Dirichlet boundary conditions (i.e. given the values of u on the boundary of U (more details are in [44])). This will help to motivate the method of solving this equation on PCF fractal domains. First write formally $\Delta G_n(x, y) = \delta^{(n)}(x - y)$, where $\delta^{(n)}$ is the n-dimensional Dirac δ-function. We say that a 'solution' to this equation is a function $G_n(x, y) : U \times U \to \mathbb{R}$ such that for each $y \in U$, $G_n(x, y)$ is twice continuously differentiable except at $x = y$, and $\Delta G_n(x, y) = 0$ when $x \neq y$ (throughout, Δ is the Laplacian in the variable $x \in \mathbb{R}^n$), and such that $\int_B \Delta G_n(x, y) d\mu(x) = 1$ for any open subset $B \subseteq U$ containing y, where μ is the Lebesgue measure. $G_n(x, y)$ is called the *fundamental solution* of Poisson's equation. First we find such a function without worrying about the boundary conditions; this can be done by using the ansatz that G_n is a function only of $||x - y||$. We can then try to find a solution H of Laplace's equation $\Delta H = 0$ throughout U with boundary conditions that cancel those of G_n, and define the *Green's function* $G(x, y) := G_n(x, y) + H(x, y)$, which will therefore vanish on the boundary of U. Note that again we can formally say that $\Delta G(x, y) = \delta^{(n)}(x - y)$. Then formally we can say that

$$\Delta \int_U G(x, y) f(y) d\mu(y) = \int_U \Delta G(x, y) f(y) d\mu(y) = \int_U \delta^{(n)}(x - y) f(y) d\mu(y) = f(x);$$

this is made more precise in [44]. Note that $\int_U G(x, y) f(y) d\mu(y)$ vanishes on the boundary of U because G does. We can then try to find a solution of Laplace's equation on u with the boundary conditions that were initially specified, and add this harmonic function to $\int_U G(x, y) f(y) d\mu(y)$ to obtain the solution of the original Poisson's equation.

This suggests that to solve Poisson's equation on a PCF fractal, it would be wise to try to find a Green's function that again formally satisfies $\Delta G(x, y) = \delta^{(n)}(x, y)$ and vanishes on the boundary of the domain. Kigami did just this for the Sierpinski gasket in [24]. In [45, pp. 46-50], Strichartz gives a different construction of the Green's function on the unit interval that takes advantage of the self-similar structure, which makes Kigami's construction seem less 'magical.' In [45, pp. 111-3], Strichartz shows that the method generalises to all PCF SSS on which there is a solution of the renormalisation problem. On the Sierpinski gasket and hexagasket, the Green's function is the pointwise limit

$$G_M(x, y) \to G(x, y)$$

as $m \to \infty$, where G_M is defined by the complicated formula

$$G_M(x,y) := \sum_{m=0}^{M} \sum_{|w|=m} r_w \sum_{z,z' \in V_1 \backslash V_0} g(z,z') \psi_z^{(1)}(x) \psi_{z'}^{(1)}(y),$$

where the constants $g(z,z')$ are defined using Δ_1 as we now explain. From (3.13) we see that for $u \in V_1 \backslash V_0$, $\Delta_1(u)$ is a linear combination of $u(z)$ for $z \in V_1$, so if $u|_{V_0}$ is zero, then Δ_1 is a linear map from $V_1 \backslash V_0$ to itself. It has trivial kernel, because if u is a nonzero on $V_1 \backslash V_0$ then without loss of generality we may assume that there exists $x \in V_1 \backslash V_0$ such that $u(x) > 0$ and $u(x)$ is maximised and there exists $y_1 \sim_1 x$ such that $u(y_1) < u(x)$, so then $\Delta_1 u(x)$ is the average difference between $u(x)$ and $u(y)$ for $y \sim_1 x$, and hence $\Delta_1 u(x) < 0$. This means that Δ_1 is an invertible linear operator on $V_1 \backslash V_0$ (if $u|_{V_0}$ is zero). For the hexagasket, for example, with respect to the ordered basis $(z, q_2, y, x, q_4, x', y', q_6, z')$ using the notation in Figure 3, the matrix is

$$\begin{pmatrix} -4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -4 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -4 \end{pmatrix}.$$

It is shown in [45] that the correct value of each entry $g(z,z')$ is $-r$ times the corresponding entry of the *inverse* of the $|V_1 \backslash V_0| \times |V_1 \backslash V_0|$ matrix corresponding to this linear map. The inverse matrix can, of course, be calculated using a tedious but straightforward algorithm.

This method then shows that for any continuous $f : K \to \mathbb{R}$, the Dirichlet problem $\Delta_\mu u = f$, $u|_{V_0} = 0$ has a solution in L given by $\int_K G(x,y) f(y) d\mu(y)$. More generally, the solution of the problem $\Delta_\mu u = f$, $u|_{V_0} = U$ for some prescribed function $U : V_0 \to \mathbb{R}$ is therefore $u = \tilde{u} + h$ where $\tilde{u}$ is the solution of $\Delta_\mu u = f$, $u|_{V_0} = 0$ and h is the harmonic function with boundary values given by U. The solution is unique because if u and v are solutions, then by the linearity of $\Delta = \Delta_\mu$ we have $\Delta(u - v) = 0$, so $u - v$ is harmonic and vanishes on V_0 and therefore $u - v \equiv 0$ so $u \equiv v$. This means that just as in the classical case, solutions of Poisson's equation are determined by their values on the boundary of the domain, which perhaps justifies the use of the term 'boundary.' This has also shown that $\Delta|_{L^0} : L^0 \to C(K)$ is an invertible linear map (in other words an isomorphism of vector spaces), where L^0 is the subspace of those functions in L that vanish on V_0. This means that L^0, and hence L, is infinite-dimensional, which shows that the definition of the Laplacian is a meaningful one. This also shows that $L = L^0 \oplus H$ where H is the $|V_0|$-dimensional space of harmonic functions on K. An interesting fact proven in [45, Corollary 2.7.5] is that if u is a non-constant function in L then u^2 is *not* an element of L. In particular this proves that L is not an algebra under pointwise multiplication. Strichartz notes in [45, p. 22] that E, on the other hand, *is* an algebra under pointwise multiplication.

In [47], Strichartz proves that if Ω is an open subset of any PCF fractal K that does not contain any boundary points, and $f : \Omega \to \mathbb{R}$ is continuous, then for any Laplacian Δ on K, the PDE $\Delta u = f$ is solvable, even if f does not extend to a continuous function on K. This is done by breaking down Ω into an infinite union of cells that overlap only at boundary points, solving an equation on each cell, and then piecing the solutions together appropriately. In the same paper, Strichartz uses the spectrum of Δ (discussed in Section 5.1) to show that on the Sierpinski gasket there are continuous functions K for which the *linear* equation $-\Delta u = \lambda u + f$ does not have any *global* solutions, which is quite different to the situation in $\mathbb{R}^n$. He also proves the analogues of the Peano and Picard-Lindelöf local existence theorems for ODEs. We can consider a system of nonlinear differential equations

$$-\Delta u = F(x, u(x)) \tag{4.2}$$

where $u(x) = (u_1(x), \dots, u_n(x))$ is a function from $K \to \mathbb{R}^n$, $F : K \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and the minus sign is conventional. Then Strichartz shows the following:

Theorem 4.2 (Local solvability)**.** *Let $A \geq 0$. Then there exists $m \in \mathbb{N}$ such that if each $a_i \in \mathbb{R}$ are such that $|a_i| \leq A$ then on any cell K_w with $|w| \geq m$ then* (4.2) *has a solution that satisfies the boundary conditions $u(F_w q_i) = a_i$ for each i.*

If in addition F is locally Lipschitz in the second variable, uniformly for the first variable (in other words for each $T > 0$ there exists $M_T > 0$ such that for all $U, U' \leq T$ and $x \in K$ we have $||F(x, U) - F(x, U')|| \leq M_T ||U - U'||$) then there is $M \in \mathbb{N}$ such that if $|a_i| \leq A$ and $|w| \geq M$ then (4.2) *has a* unique *solution on K_w that satisfies $u(F_w q_i) = a_i$ for each i.*

As in the case of ODEs, under the locally Lipschitz assumption, existence and uniqueness are proved in [47] using Banach's contraction mapping theorem. Strichartz states in [47] that "standard arguments show that one may reduce higher order equations to the form (4.2) by introducing new variables."

5 Linear Heat Equation

From now on we will be studying linear and semilinear heat equations on fractal domains, which are used to study diffusion processes on such domains. Such processes are very common in nature, modelling a situation in which a large number of small particles immersed in a fluid or conducting system undergo a seemingly random motion as a result of frequent collisions (see [42] for a summary). The simplest form of the heat equation is $\partial u/\partial t = c\Delta u$ for $t > 0$ where $u(t, x)$ is the concentration of such particles (which depends on time t and spatial position x), c is a constant, and Δ is some kind of Laplacian, which (roughly) measures the rate at which the average value of a function over small balls of small radius r deviates from the value of the function at that point as r grows. The diffusion equation therefore says that at points of higher concentration than the surroundings (where the Laplacian will be negative), the concentration will decrease with time, which makes intuitive sense. It can be used to model how the temperature profile of an object changes with time, given an initial temperature profile, but there are many other applications of diffusion processes in the

natural sciences. It is noted in [13] that "applications of diffusion include sintering, i.e. making solid materials from powder (powder metallurgy, production of ceramics); catalyst design in the chemical industry; diffusion of steel (e.g. with carbon of nitrogen) to modify its properties; doping during production of semiconductors." One of the earlier examples of the use of diffusion processes on fractals in particular is an interesting 1982 study involving percolation theory by Rammal and Toulouse [40].

5.1 Bounded fractal domains

We first consider the linear heat equation

$$\partial u/\partial t = \Delta u \tag{5.1}$$

for $t > 0$ on bounded fractal domains, letting the constant $c = 1$ for convenience. Given a linear operator like the Laplacian on a PCF SSS like the hexagasket H, a natural thing to consider is its spectrum, and indeed this will prove useful when considering the linear heat equation. Kigami [25] proves an analogue of the Gauss-Green formula (Green's second identity) on an open $\Omega \subset \mathbb{R}^n$

$$\int_\Omega (u\Delta v - v\Delta u) = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS$$

on a broad class of PCF fractals (including the examples that we are interested in). Note that an analogue of this result requires an analogue of normal derivatives to be defined on K. It follows from this result that the Dirichlet Laplacian, i.e. Δ restricted to the space L^0 of functions in L vanishing on the boundary, is a self-adjoint operator, so its eigenvalues are all real. Kigami also shows in [25, Corollary 3.4.7] that if the self-similar structure of a PCF fractal satisfies some technical conditions (which hold for H and the Sierpinski gasket, for example) then the Dirichlet Laplacian has compact resolvent, and so by the spectral theorem has a complete orthonormal basis of eigenfunctions $u_j \in L^0$ with corresponding eigenvalues $\lambda_j \in \mathbb{R}$ such that

$$-\Delta u_j = \lambda u_j.$$

The eigenvalues can be computed using a procedure known as *spectral decimatiion*, where a quadratic function is repeatedly iterated. In [45, Chapter 3], Strichartz carries out this procedure for the Sierpinski gasket and shows that an analogous procedure can recover the familiar eigenvalues $\pi^2 k^2$ for $k = 1, 2, 3, \ldots$ of the Laplacian (second derivative) on the unit interval, corresponding to eigenfunctions $\sin(\pi k x)$, with Dirichlet boundary contitions $u(0) = u(1) = 0$. Spectral decimation only works for quite a limited class of PCF SSS (see [43]), but it does work for the hexagasket (see [50]).

The asymptotic behaviour of Dirichlet eigenvalues has been investigated. Let $\rho(x)$ be the number of Dirichlet eigenvalues less than or equal to $x \in \mathbb{R}$. On open, bounded, connected, subsets of $\mathbb{R}^n$, a famous theorem known as Weyl's theorem implies that $\rho(x)/x^{n/2}$ converges to a positive, finite limit as $x \to \infty$ [16, 33, 34]. In [26], Kigami and Lapidus prove an analogue of this result on fractal domains, showing that the quantity that takes the place of n is not the Hausdorff dimension, as might be expected, but another quantity called the spectral dimension.

Theorem 5.1. *Consider a self-similar, regular probability measure on a PCF SSS with a regular harmonic structure whose energy has renormalisation factors $r_i \in (0,1)$ as in* (3.12) *(for example H or the Sierpinski gasket). There are constants $0 < c < C$ and $M > 0$ such that if $x > M$ then*

$$cx^{d_s/2} \leq \rho(x) \leq Cx^{d_s/2}, \tag{5.2}$$

where d_s, called the spectral dimension *associated with the structure, is the unique positive number such that*

$$\sum_i (r_i \mu_i)^{d_s/2} = 1. \tag{5.3}$$

In this case, however, $\rho(x)/x^{n/2}$ does not generally converge as $x \to \infty$ (see [43]). Note that (5.2) means that

$$d_s = \lim_{x\to\infty} \frac{2\log(\rho(x))}{\log x}.$$

For the hexagasket H with the standard measure and self-similar energy with equal weights, for example, we have

$$\begin{aligned} \sum_i \left(\frac{3}{7}\cdot\frac{1}{6}\right)^{d_s/2} &= 1 \\ \left(\frac{1}{14}\right)^{d_s/2} &= \frac{1}{6} \\ d_s &= \frac{2\log 6}{\log 14} \approx 1.36 \end{aligned}$$

Using the complete orthonormal basis of eigenfunctions, Strichartz [45, p. 149] formally writes the solution of (5.1) with initial condition $u(0,x) = \phi(x)$ and boundary conditions $u(t,q_i) = 0$ for all i as

$$u(t,x) = e^{t\Delta}\phi(x) = \sum_j e^{-t\lambda_j} \left(\int_K f(y)u_j(y)d\mu(y)\right) u_j(x)$$

as in Parseval's identity, and explains that the sum and integral can often be interchanged to give

$$u(t,x) = \int_K k(t,x,y)f(y)d\mu(y)$$

where

$$k(t,x,y) = \sum_j e^{-t\lambda_j} u_j(x)u_j(y)$$

is the *heat kernel*. Note that if x is a boundary vertex then $u_j(x) = 0$ so $k(t,x,y) = 0$ for all $y \in K, t > 0$, so $u(t,x) = 0$ for all $t > 0$. Note also that k is symmetric in x and y.

5.2 Unbounded fractal domains

We now consider the linear heat equation on unbounded fractal domains. Using the same notation as in (2.1) for the unbounded hexagasket, note that $\Gamma_1 = \Gamma_1^{(0)}$ of $H = H_0$ is a subset of $\Gamma_2^{(1)}$ of the

larger hexagasket H_1, which is a subset of $\Gamma_3^{(2)}$ of the larger hexagasket H_2 and so on. Generally, $\Gamma_m^{(j)}$ of H_j is a subset of $\Gamma_{m+k}^{(j+k)}$ of H_{j+k} for $m \geq 1, j, k \geq 0$. Therefore we can define

$$\Gamma_m^{(\infty)} = \bigcup_k \Gamma_{m+k}^{(k)}$$

for the unbounded hexagasket. We can then define Brownian motion on the unbounded hexagasket just as we did for the bounded hexagasket in Section 3.5, and the analysis in that section still holds in this context. We can define a Laplacian on the unbounded hexagasket using (3.18) or (3.19). We have seen that $X_m(t)$, and therefore in the limit $X(t)$, takes on average time $14\alpha_m = 14^{-m+1}$ to move between adjacent vertices of V_{m-1}, which are 3^{-m} apart, so $X(t)$ is likely to move a distance of order $h^{\log 3/\log 14}$ in a time interval h. The *walk dimension* will therefore be defined as

$$d_w = \frac{\log 14}{\log 3} \approx 2.40$$

for H. There are constants $0 < c < C$ such that $ch^{2/d_w} \leq \mathbb{E}(||X(t+h) - X(t)||^2) \leq Ch^{2/d_w}$, in analogy with $\mathbb{E}(||X(t+h) - X(t)||^2) = h$ for standard Brownian motion on $\mathbb{R}^n$, where the walk dimension is 2 (independent of n). This was proved for the example of the unbounded Sierpinski gasket in [1, 40]. The Hausdorff dimension depends only on the set itself, and is therefore a purely geometrical property; the walk and spectral dimensions also depend on the self-similar structure, regular probability measure, and harmonic structure, so they are analytical, rather than purely geometrical, properties (see [26]). From the definitions that we have given, it is not clear that there is a relation between the Hausdorff, spectral and walk dimensions, but in fact we have

$$\frac{d_s}{2} = \frac{d_f}{d_w}$$

in general [15], and it can be verified directly for the hexagasket:

$$\frac{2\log 6}{\log 14}\Big/2 = \frac{\log 6}{\log 3}\Big/\frac{\log 14}{\log 3}\ .$$

This is useful, as it means that if we calculate two of these quantities for a self-similar structure, we get the third for free.

Brownian motion can be used to define a *heat kernel* or *transition density* $k(t, x, y)$ to be the probability density measuring the likelihood that $X(t)$ is at position y after time t if it started at position x:

$$\mathbb{P}(X(t_0 + t) \in A | X(t_0) = x) = \int_A k(t, x, y) d\mu(y)$$

for any measurable set A and any fixed time t_0. This should always be positive for all $t > 0, x, y \in \mathbb{R}^n$ and normalised with $\int_{\mathbb{R}^n} k(t, x, y) d\mu(y) = 1$. The random walk that is the Brownian motion is a Markov process, so the increments $X(t+h) - X(t)$ for $t, h > 0$ are independent. If $s, t > 0$ and $X(s+t) = y$ then X must have been at some position $z \in \mathbb{R}^n$ at time s, so we can integrate over such z to give the 'semigroup property' $k(s+t, x, y) = \int_{\mathbb{R}^n} k(t, x, z)k(s, z, y)d\mu(z)$. In the case of

standard Brownian motion on $\mathbb{R}^n$, the increments are also normally distributed, and the heat kernel is the Gaussian

$$k(t,x,y) = (2\pi t)^{-n/2} \exp\left(-\frac{||x-y||^2}{2t}\right). \tag{5.4}$$

The constant $(2\pi)^{-n/2}$ has been chosen so that k satisfies the normalisation condition that for all $t > 0$, k integrates to 1 by a double integral argument. Note also that k is a function of t and $r := ||x-y||$ only (as it must be by symmetry), so is symmetric in x and y. Differentiating using the product and chain rules, we see that if we fix $y \in \mathbb{R}^n$ and Δ is the Laplacian with respect to the x variable, then

$$\frac{\partial k}{\partial t} = \left(-\frac{n}{2}(2\pi t)^{-n/2-1} + (2\pi t)^{-n/2}\frac{||x-y||^2}{2t^2}\right)\exp\left(-\frac{||x-y||^2}{2t}\right) = \frac{1}{2}\Delta k,$$

so k satisfies this linear heat equation. The initial condition is $k(t,x,y) \to \delta^{(n)}(x-y)$ as $t \to 0^+$ in the sense of distributions: for any smooth function $\psi : \mathbb{R}^n \to \mathbb{R}$ and $x \in \mathbb{R}^n$ we have $\int_{\mathbb{R}^n} k(t,x,y)\psi(y)d\mu(y) \to \psi(x)$ as $t \to 0^+$. The boundary condition is that $k(t,x,y) \to \infty$ as $||x|| \to \infty$ for each $t > 0$ and $y \in \mathbb{R}^n$. Thus $k(t,x,y)$ satisfies an 'approximate identity' condition, and can be regarded as the temperature distribution at time t and position x resulting from an initial condition of a point source of particles at y following independent Brownian paths. Similarly, for a different initial condition, given by $\phi : \mathbb{R}^n \to \mathbb{R}$, Falconer states in [11, p. 241], "diffusion of heat on $\mathbb{R}^n$ may be thought of as the aggregate effect of a large number of 'heat particles' following independent Brownian paths." We can define

$$u(t,x) = \int_{\mathbb{R}^n} k(t,x,y)\phi(y)d\mu(y)$$

where μ is the Lebesgue measure. It can be shown that the order of integration and differentiation can be exchanged, so u also satisfies the linear heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u,$$

and the approximate identity condition for the heat in a measurable set A at small times: $\int_A u(t,x)dx \to \int_A \phi(x)d\mu(x)$ as $t \to 0^+$. Thus $u(t,x)$ models the temperature at time t and point x if the initial temperature distribution is given by ϕ.

Heat kernels can be defined similarly for unbounded fractals like the unbounded hexagasket on which there is a Brownian motion $X(t)$. If we define the discrete heat kernel $k_m(n\alpha_m, x, y) = \mathbb{P}^x(X(n\alpha_m) = y)$ for $m, n \in \mathbb{N}$, $x, y \in \Gamma_m$, then k_m satisfies a discrete heat equation

$$\begin{aligned}\tilde{\Delta}k_m(n\alpha_m, x_0, x) &= \frac{1}{\mu_0(x)}\sum_{y\sim x}(k_m(n\alpha_m, x_0, y) - k_m(n\alpha_m, x_0, x)) \\ &= \frac{1}{\mu_0(x)}\sum_{y\sim x} k_m(n\alpha_m, x_0, y) - k_m(n\alpha_m, x_0, x) \\ &= k_m((n+1)\alpha_m, x_0, x) - k_m(n\alpha_m, x_0, x),\end{aligned}$$

where the Laplacian is in the x variable, and $x_0 \in \Gamma_n$ is fixed. Dividing through by α_m, taking the limit $m \to \infty$ and using (3.18) suggests that the heat kernel $k(t, x, y)$ of the limit process $X(t)$ satisfies the heat equation

$$\frac{\partial k}{\partial t} = \frac{1}{6}\Delta k.$$

The constant $1/6$ is inconsequential; scaling time $t \to ct$ for an appropriate constant c will eliminate it. It is the relationship between the constants 14 in the definition of the pointwise Laplacian and $d_w = \log 14/\log 3$ that lead to this non-degenerate limit. The kernel k will satisfy many of the same properties as the Gaussian kernel; positivity, normalisation, and the semigroup property will all hold for all the same reasons. It can be shown by analysing $X(t)$ that in cases such as the Brownian motion that we have constructed on the unbounded hexagasket, an appropriate approximate identity condition will hold and there will be symmetry in x and y. However, the fractality of the domain means that obtaining an exact formula like (5.4) is usually impossible, and obtaining useful upper and lower bounds is an area of active research. There will usually be exponential decay with the distance between the spacial arguments, and an inverse power law decay with time. A way to try to obtain such bounds is to verify by substitution that the Gaussian kernel (5.4) satisfies a second-order PDE called the *standard diffusion equation* (see [41])

$$\frac{\partial k}{\partial t} = \frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial k}{\partial r}\right).$$

This can be adapted when the walk dimension is d_w to the *standard diffusion equation on fractals* (see [36])

$$\frac{\partial k}{\partial t} = \frac{1}{r^{d_f-1}}\frac{\partial}{\partial r}\left(r^{d_f-1}r^{2-d_w}\frac{\partial k}{\partial r}\right)$$

whose exact solution can be verified by substitution to be

$$k(t, r) = \frac{A_{d_s}}{t^{d_s/2}}\exp\left(-\frac{r^{d_w}}{d_w^2 t}\right)$$

for some constant A_{d_s}. This is a good approximation to the heat kernel when r is small relative to the t, but not when $r \gg t^{1/d_w}$. In [42], Roman discusses how to obtain estimates for k in the case $r \gg t^{1/d_w}$ using fractional derivatives.

We now consider a more general setting, in preparation for analysing the semilinear heat equation in Chapter 6, where an appropriate nonlinear term $f(x, u)$ is added to the right-hand side of the linear heat equation. The setup for this section will be similar to the introduction of [12] (though in a bit more detail), as the objective of the next section will be to study more general semilinear equations than those studied in that article. Let G be an unbounded, Borel, locally compact subset of $\mathbb{R}^n, n > 1$, (recalling that *locally compact* means that every point of G has a compact neighbourhood). Often, G will be a fractal with walk dimension $d_w > 1$. Assume that G admits a locally finite Borel measure μ; henceforth, integration and L^p spaces will always be with respect to the measure μ. An archetypal example of such a space is the unbounded hexagasket (part of which is shown in Figure 4), with μ being (a constant multiple of) the restriction of the $\log 6/\log 3$-dimensional

Hausdorff measure to G. We study the linear heat equation

$$\frac{\partial u}{\partial t} = \Delta u, \quad t > 0 \tag{5.5}$$

on G for appropriate measurable functions $u : (0, \infty) \times G \to \mathbb{R}$, with initial data

$$u(0, x) = \phi(x), \quad x \in G$$

for some non-negative, measurable $\phi : G \to \mathbb{R}$.

As we have seen, there are many different definitions of the Laplacian on fractal sets, so (5.5) and (6.1) only make sense if we specify which. Instead of defining the Laplacian using graph approximations or Dirichlet forms, we will now define it in terms of a heat kernel, the existence of which will be a key assumption. But previously we defined heat kernels using a Laplacian or Brownian motion, neither of which we will assume a priori to exist on G, so we now give a stand-alone definition of a heat kernel as a function that satisfies many of the properties that we have already seen are satisfied by the heat kernels we have seen on particular fractal domains.

Definition 5.1. A *heat kernel* on G is a continuous function $k : (0, \infty) \times G \times G \to \mathbb{R}$ that satisfies conditions (K1)-(K5), following [12]:

(K1) (Positivity) $k(t, x, y) > 0$ for all $x, y \in G, t > 0$;

(K2) (Symmetry) $k(t, x, y) = k(t, y, x)$ for all $x, y \in G, t > 0$;

(K3) (Normalisation) $\int_G k(t, x, y) d\mu(y) = 1$ for all $x \in G, t > 0$;

(K4) (Semigroup property) $k(s + t, x, y) = \int_G k(t, x, z) k(s, z, y) d\mu(z)$;

(K5) (Approximate identity) $\int_G k(t, x, y) f(y) d\mu(y) \to f(x)$ as $t \to 0^+$, in the L^2-norm for all $f \in L^2(G)$.

These properties mean that it still makes sense to think of $k(t, x, y)$ as the probability density that a particle undergoing an appropriate random walk on G will be at position y after time t if it starts at position x at time 0; keeping this in mind makes the conditions seem more natural. There is some variation in the definition of heat kernels in different settings. In the setting of metric measure spaces, for example in [13, 19], (K5) is omitted and continuity is not required, only joint measurability in x and y of $k(t, \cdot, \cdot) : G \times G \to \mathbb{R}$ for each $t > 0$. Nor is the normalisation condition required (this is called the *conservative* property), merely the inequality $\int_G k(t, x, y) d\mu(y) \leq 1$ for all $x \in G, t > 0$.

Often, it will be necessary to assume that the heat kernel satisfies estimates of the following form:

(K6) (Bounds) There exist constants $0 < c_1 \leq c_2$ and $a_1, a_2 > 0$ and $d_s, d_f > 0, d_w > 1$ such that $d_s/2 = d_f/d_w$ and such that for all $t > 0$ and $x, y \in G$ we have

$$a_1 t^{-d_s/2} \exp\left(-c_1 \left(\frac{||x - y||^{d_w}}{t}\right)^{\frac{1}{d_w - 1}}\right) \leq k(t, x, y) \leq a_2 t^{-d_s/2} \exp\left(-c_2 \left(\frac{||x - y||^{d_w}}{t}\right)^{\frac{1}{d_w - 1}}\right).$$

If G has Hausdorff, spectral and walk dimensions and their values are d_f, d_s and d_w respectively then it is plausible that (K6) will indeed hold, hence the choice of names for these constants.

This does indeed hold for the unbounded hexagasket (to which the theory of this section and the next chapter will apply). Recall that for the hexagasket H, we have $d_f = \log 6/\log 3 \approx 1.63$, $d_w = \log 14/\log 3 \approx 2.40$ and $d_s = 2\log 6/\log 14 \approx 1.36$, related by the formula $d_f/d_s = d_w/2$.

For regularity results related to Hölder continuity of the solution, a Hölder condition on the heat kernel will also be required. This will certainly be satisfied by the Gaussian kernel (5.4).

(K7) (Hölder condition) There exist $M_0 > 0$, $\nu \geq 1$ and $0 < \sigma \leq 1$ such that for all $t > 0$ and $x_1, x_2, y \in G$ we have

$$|k(t, x_2, y) - k(t, x_1, y)| \leq M_0 t^{-\nu} ||x_2 - x_1||^{\sigma}.$$

One of the classes of fractals that heat kernels have been constructed on are nested fractals (as defined in Definition 2.18), a class that includes the Sierpinski gasket and the hexagasket. On both bounded and unbounded nested fractals, heat kernels have been constructed which satisfy (K1)-(K6) in [15]. In addition, in [4] it is shown that (K7) holds for the Sierpinski gasket with $\nu = 1$ and $\sigma = d_w - d_f$. Therefore, all the results in the remainder of this chapter and the next chapter will hold for the unbounded Sierpinski gasket, and all except possibly Theorem 6.4 will hold for the unbounded hexagasket. The existence and properties of heat kernels on other classes of fractals is a nontrivial problem.

Next, we define a family $\{P_t | t > 0\}$ of integral operators associated with k by

$$P_t g(x) = \int_G k(t, x, y) g(y) d\mu(y). \tag{5.6}$$

Using the interpretation of k as a probability density associated with a random walk, it makes intuitive sense that $P_t\phi$ is the solution of the linear heat equation with initial data ϕ; the integral in (5.6) is summing up (over all $y \in G$) the amount of heat that has been transferred from position y to position x after time t. A key property of the family P_t is the contraction property. This is stated but not proved in [12], and the following proof is the result of discussions with Prof. Kenneth Falconer.

Proposition 5.2. *[Contraction property] Assume that k satisfies (K1)-(K5), and let $1 \leq q \leq \infty$. Then for all $g \in L^q(G)$ and $t > 0$ we have $||P_t g||_q \leq ||g||_q$.*

Proof. First consider the case $q = \infty$. For each $x \in G$,

$$|P_t g(x)| = \left| \int_G k(t, x, y) g(y) d\mu(y) \right| \leq \int_G |k(t, x, y) g(y)| d\mu(y) \leq \int_G k(t, x, y) d\mu(y) ||g||_\infty = ||g||_\infty$$

so $||P_t g||_\infty \leq ||g||_\infty$.

Now let $1 < q < \infty$. Let $p > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Hölder's inequality gives that

$$\begin{aligned}
\int_G k(t, x, y) |g(y)| d\mu(y) &= \int_G k(t, x, y)^{1/p} k(t, x, y)^{1/q} |g(y)| d\mu(y) \\
&\leq \left[\int_G k(t, x, y) d\mu(y) \right]^{1/p} \left[\int_G k(t, x, y) |g(y)|^q d\mu(y) \right]^{1/q} \\
&= \left[\int_G k(t, x, y) |g(y)|^q d\mu(y) \right]^{1/q}.
\end{aligned}$$

This inequality

$$\int_G k(t,x,y)|g(y)|d\mu(y) \leq \left[\int_G k(t,x,y)|g(y)|^q d\mu(y)\right]^{1/q}$$

also holds in the case $q = 1$, trivially. Therefore if $1 \leq q < \infty$ then

$$\begin{aligned} ||P_t g||_q^q &= \int_G \left|\int_G k(t,x,y)g(y)d\mu(y)\right|^q d\mu(x) \\ &\leq \int_G \left[\int_G k(t,x,y)|g(y)|d\mu(y)\right]^q d\mu(x) \\ &\leq \int_G \int_G k(t,x,y)|g(y)|^q d\mu(y)d\mu(x) \\ &= \int_G |g(y)|^q d\mu(y) \qquad (5.7) \\ &= ||g||_q^q, \end{aligned}$$

where (5.7) follows from (K3) upon switching the order of integration; this is valid by Fubini's theorem, since $||g||_q^q < \infty$ as $g \in L^q(G)$. The result follows.

□

In particular, this is a family of operators on L^2 with the following property, again stated but not proved in [12]:

Proposition 5.3 (Semigroup property)**.** *Assume that k satisfies (K1)-(K5). Then if $s, t > 0$ then $P_{s+t} = P_s \circ P_t$.*

Proof. Note first that μ is assumed to be locally finite, hence σ-finite. Suppose $g \in L^2(G) \cap L^1(G)$. Then by the contraction property, $P_{s+t}g \in L^2(G)$. Therefore, for μ-almost all $x \in G$ we have that $P_{s+t}g(x) < \infty$, so by (K4) and Fubini's theorem we have that

$$\begin{aligned} P_{s+t}g(x) &= \int_G k(s+t,x,y)g(y)d\mu(y) \\ &= \int_G \int_G k(s,x,z)k(t,z,y)d\mu(z)g(y)d\mu(y) \\ &= \int_G \int_G k(s,x,z)k(t,z,y)g(y)d\mu(y)d\mu(z) \\ &= \int_G k(s,x,z)P_t g(z)d\mu(z) \\ &= (P_s(P_t g))(x) \end{aligned}$$

for μ-almost all $x \in G$. Therefore $P_{s+t}g = (P_s \circ P_t)g$. □

(K5) is equivalent to the statement that this family of operators is continuous in the L^2-norm, i.e. $||P_t g - g||_2 \to 0$ as $t \to 0^+$. This suggests that it is reasonable to define an infinitesimal generator; this will be the definition of the Laplacian Δ in this context:

$$\Delta g = \lim_{t \to 0^+} \frac{P_t g - g}{t}$$

where g is any function in $L^2(G)$ such that the above limit exists in the L^2 norm and is finite. This is the definition we will use when referring to the heat equation (5.5). Note that if the heat kernel is the transition density of an analogue X of Brownian motion on G, then $P_t f(x) = \mathbb{E}^x(f(X(t)))$, so this definition coincides with (3.19). In fact, whenever a domain has a Laplacian Δ that is a bounded, negative-definite, self-adjoint operator on a Hilbert space of functions, Fukushima [17] shows that the family $\{P_t = e^{(t\Delta)}, t > 0\}$ of operators forms a strongly continuous semigroup, which will often have many of the same properties as P_t. This shows how intimately connected Laplacians, Brownian motion and heat kernels are.

Now we prove a result that ties together the definitions of P_t, Δ and the heat equation (5.5):

Theorem 5.4. *Suppose that the heat kernel k satisfies (K1)-(K5). Let $T > 0$ and suppose that the initial data ϕ is in $L^2(G)$. Note that by the contraction property, for each $t > 0$, $P_t\phi \in L^2(G)$. Suppose that $\frac{\partial}{\partial t} P_t\phi(x)$ exists for all $t \in (0,T), x \in G$, and that for each $t_0 \in (0,T)$, $(P_{t_0+t}\phi - P_t\phi)/t$ converges to a finite limit in the L^2 norm as $t \to 0^+$. Then $P_t\phi$ is a strong solution of (5.5) on $(0,T) \times G$.*

Proof. Let $t_0 \in (0,T)$ and $t > 0$ such that $t_0 + t \in (0,T)$. For each $x \in G$,

$$\frac{P_{t_0}P_t\phi(x) - P_t\phi(x)}{t} = \frac{P_{t_0+t}\phi(x) - P_t\phi(x)}{t} \xrightarrow[t\to 0^+]{} \frac{\partial}{\partial t} P_t\phi(x)\Big|_{t=t_0}.$$

Since the limit is also assumed to exist in the L^2 norm (and be equal to $\Delta P_t\phi$ by the definition of Δ), this limit must equal the pointwise limit $\frac{\partial}{\partial t} P_t\phi(x)\Big|_{t=t_0}$, so $P_t\phi$ is a (strong) solution of (5.5). □

If the existence of the L^2 limit of $(P_{t_0+t}\phi - P_t\phi)/t$ in the L^2 norm is difficult to verify directly, it is sufficient to prove that the dominating function $G \to \mathbb{R}$ given by $x \mapsto \sup\left\{\left|\frac{P_t\phi(x) - P_s\phi(x)}{t-s}\right| \middle| t, s \in (0,T)\right\}$ is an element of $L^2(G)$. Indeed, if this is the case and $(P_{t_0+t}\phi - P_{t_0}\phi)/t \to \frac{\partial}{\partial t} P_t\phi(x)\Big|_{t=t_0}$ pointwise as $t \to 0^+$ but not in L^2 then there is a subsequence $(t_n)_{n\geq 1}$ such that $((P_{t_0+t_n}\phi - P_{t_0}\phi)/t_n)^2 \to \left(\frac{\partial}{\partial t} P_t\phi(x)\big|_{t=t_0}\right)^2$ pointwise as $t \to 0^+$ but not in L^1; by the square integrability of the dominating function, this contradicts the dominated convergence theorem.

We now state a regularity result that says that under a technical assumption μ, if k and ϕ satisfy certain Hölder conditions and and the initial data is bounded in L^1 and L^∞ then P_t is Hölder continuous in x. This will be used to prove Theorem 6.4 about Hölder continuity of solutions of semilinear equations.

Theorem 5.5. *[Hölder continuity]*

Suppose that the measure μ satisfies the condition

$$\exists m > 0 \quad \forall r > 0\, \forall x \in G \quad \mu(B_r(x)) \leq m r^{d_f} \qquad (*)$$

where d_f is the same constant as in (K6) (recall that this holds if G is self-similar with Hausdorff dimension d_f and μ is the restriction of d_f-dimensional Hausdorff measure to G). Assume that k

satisfies (K1)-(K7). Assume that $\phi \in L^1(G) \cap L^\infty(G)$ is Hölder continuous with exponent $\lambda \in (0,1]$, i.e. there exists $C_0 > 0$ such that for all $x_1, x_2 \in G$ we have

$$|\phi(x_2) - \phi(x_1)| \leq C_0 ||x_2 - x_1||^\lambda.$$

Then for any $T > 0$, $P_t\phi$ is Hölder continuous in x with exponent $\gamma := \lambda\sigma/(\lambda + \nu d_w)$, uniformly for $t \in (0,T)$, where d_w is as in (K6) (usually the walk dimension) and ν is as in (K7). In other words, there is $C_1 > 0$ depending on T such that for all $t \in (0,T)$ for all $x_1, x_2 \in G$ we have

$$|P_t\phi(x_2) - P_t\phi(x_1)| \leq C_1 ||x_2 - x_1||^\gamma.$$

Proof outline. By the contraction property we have $||P_t\phi||_1 \leq ||\phi||_1$ and $||P_t\phi||_\infty \leq ||\phi||_\infty$ for all $t > 0$. The proof uses two technical results, [12, Proposition 4.1 and Corollary 4.2] that use condition $(*)$ on the measure to bound certain integrals on G. The proof then follows the start of [12, Theorem 4.3]. The idea is to choose an appropriate $s > 0$ and then break down the inequality into two cases depending on whether $t \geq ||x_2 - x_1||^s$ or $t < ||x_2 - x_1||^s$. The former case uses the integrability of ϕ and (K7) and while the second case uses the upper bound of (K6), the Hölder condition on ϕ, and the corollary of condition $(*)$. This method gives the required Hölder condition when $||x_2 - x_1|| < 1$. As $||P_t\phi||$ is uniformly bounded, the condition holds for all $x_1, x_2 \in G$ after increasing C_1, as required. □

6 Semilinear Heat Equation

The semilinear heat equation $\partial u/\partial t = \Delta u + f(u)$ for suitable functions f on *bounded* fractal domains is studied in [22]. Throughout this chapter, the domain under consideration will be *unbounded*. Let the set $G \subseteq \mathbb{R}^n$ and μ be as in Section 5.2 (again, the unbounded hexagasket with the restriction of $\log 6/\log 3$-dimensional Hausdorff measure is an archetypal example). We now consider the semilinear heat equation

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + f(x,u), \quad t > 0 \tag{6.1}$$

on G for appropriate measurable functions $f : (0,\infty) \times G \to \mathbb{R}$, with non-negative initial values

$$u(0,x) = \phi(x), \quad x \in G.$$

This equation was considered in [12] for the particular case $f(x,u) = u^p$ for different values $p > 1$, and we now generalise the results to other functions $f(x,u)$, usually following similar methods of proof. As in Section 5.2, the main assumption will be the existence of a heat kernel satisfying several conditions; the same notation will be used to represent these conditions.

As we have seen previously, when studying solutions of differential equations on fractals, it is often useful to consider a weak formulation, and this is what we will do for the remainder of this chapter. Henceforth, by a 'solution' to the semilinear heat equation, we will mean a measurable function $u : (0,\infty) \times G \to \mathbb{R}$ or $u : (0,T) \times G \to \mathbb{R}$ for some $T \in (0,\infty)$ satisfying the integral

equation

$$u(t,x) = \int_G k(t,x,y)\phi(y)d\mu(y) + \int_0^t d\tau \int_G k(t-\tau,x,y)f(x,u(\tau,x))d\mu(y). \tag{6.2}$$

In this vein, the *definition* 5.6 of $P_t\phi$ may be considered as the weak formulation of the *linear* equation (5.5); in this case, when f is identically zero, Theorem 5.4 demonstrates the link between the weak and strong solutions. Under certain conditions it should be possible to show that sufficiently regular solutions of (6.2) are also solutions of (6.1); [12, Theorem 4.5] attempts to prove such a result in the case when $f(x,u) = u^p$, but omits many details. We say that a solution u is *global* if $u : (0,\infty) \times G \to \mathbb{R}$.

A physical application may be that (6.1) can be thought of as a nonlinear reaction-diffusion equation. Therefore solutions may model the evolution with time of temperature u on a large fractal catalyst, where the nonlinear term $f(u)$ results from a change in temperature due to a chemical reaction whose rate changes with temperature and, possibly, location on the catalyst.

6.1 Non-existence

We now show using the lower bound of the assumed heat kernel estimate that there are no non-negative global solutions for any nonzero, non-negative initial data, however small, provided that f satisfies a certain condition. What happens is that the L^∞ norm of the solution approaches infinity in finite time, a phenomenon knows as *blow-up* that is familiar in PDE theory. In this section, we will see that the non-negativity of ϕ, u, f will be of crucial importance. A key tool in the proof will be the following estimate. The proposition and proof are based on [49, Theorem 5] (a specific case is [12, Proposition 2.1]) but with a bit more detail, and adapted to consider a more general nonlinear term.

Proposition 6.1. *Assume that the heat kernel k satisfies (K1)-(K4) (noting that (K5) and (K6) are not required here). Let $p > 1$ and suppose that there is a constant $D > 0$ such that for all $x \in G, U \geq 0$ we have $f(x,U) \geq DU^p$. Then there is a constant M_1 such that for any essentially bounded initial data $\phi, T > 0$, and non-negative essentially bounded solution $u(t,x)$ on $(0,T) \times G$, the estimate*

$$t^{\frac{1}{p-1}} \int_G k(t,x,y)\phi(y)d\mu(y) \leq M_1 \tag{6.3}$$

holds for all $t \in (0,T), x \in G$.

There are two things about the above condition on f that deserve emphasis: firstly, the uniformity of the constant D in $x \in G$, and secondly the fact that $f(x,U) = U^p$ is a canonical example of such an f.

Proof. Note firstly that if $\frac{1}{p} + \frac{1}{q} = 1$ then for any $t, \tau > 0, x \in G$, the weighted Hölder inequality (i.e. Hölder's inequality with the measure that gives a Borel set A weight $\int_A k(t-\tau,x,y)d\mu(y)$) applied to the functions 1 and $P_\tau\phi$ gives

$$\int_G k(t-\tau,x,y)P_\tau\phi(y)d\mu(y) \leq \left(\int_G k(t-\tau,x,y)[P_\tau\phi(y)]^p d\mu(y)\right)^{1/p} \times 1^{1/q}. \tag{6.4}$$

Because of the non-negativity assumptions on f and u, from (6.2) we have

$$u(t,x) \geq P_t\phi(x).$$

Applying the above, and applying (6.2) again, we have

$$\begin{aligned}
u(t,x) &\geq \int_0^t d\tau \int_G k(t-\tau,x,y) f(y,u(\tau,y)) d\mu(y) \\
&\geq \int_0^t d\tau \int_G k(t-\tau,x,y) D[u(\tau,y)]^p d\mu(y) \\
&\geq \int_0^t d\tau \int_G k(t-\tau,x,y) D[P_\tau\phi(y)]^p d\mu(y) \\
&\geq D\int_0^t d\tau \left(\int_G k(t-\tau,x,y) P_\tau\phi(y) d\mu(y)\right)^p \\
&= D\int_0^t d\tau [P_{t-\tau}(P_\tau\phi)(x)]^p \\
&= D\int_0^t d\tau [P_t\phi(x)]^p \\
&= Dt[P_t\phi(x)]^p.
\end{aligned}$$

We claim that for each $n \in \mathbb{N}$ we have

$$u(t,x) \geq (Dt)^{1+p+\cdots+p^{n-1}} [P_t\phi(x)]^{p^n} / g_n(p) \tag{6.5}$$

where

$$g_n(p) := (1+p)^{p^{n-2}} (1+p+p^2)^{p^{n-3}} \cdots (1+p+\cdots+p^{n-1}).$$

The justification of this (not written in [12] or [49]) is as follows: we have just shown this to be true for $n = 1$. Assuming it is true for n, then substituting this inequality as above, we have

$$\begin{aligned}
u(t,x) &\geq \int_0^t d\tau \int_G k(t-\tau,x,y) f(y,u(\tau,y)) d\mu(y) \\
&\geq \int_0^t d\tau \int_G k(t-\tau,x,y) D[u(\tau,y)]^p d\mu(y) \\
&\geq \int_0^t d\tau \int_G k(t-\tau,x,y) D[(D\tau)^{1+p+\cdots+p^{n-1}} [P_\tau\phi(x)]^{p^n} / g_n(p)]^p d\mu(y) \\
&\geq D\int_0^t d\tau \int_G k(t-\tau,x,y) ((D\tau)^{1+p+\cdots+p^{n-1}})^p [P_\tau\phi(x)]^{p^{n+1}} / (g_n(p))^p d\mu(y) \\
&= D(D^{1+p+\cdots+p^{n-1}})^p \int_0^t \tau^{p+p^2+\cdots+p^n} [P_t\phi(x)]^{p^{n+1}} / (g_n(p))^p d\tau \\
&= D^{1+p+\cdots+p^n} t^{1+p+\cdots+p^n} [P_t\phi(x)]^{p^{n+1}} / (g_n(p))^p (1+p+\cdots+p^n) \\
&= (Dt)^{1+p+\cdots+p^n} [P_t\phi(x)]^{p^{n+1}} / g_{n+1}(p).
\end{aligned}$$

So by induction (6.5) holds for all $n \in \mathbb{N}$. Raising both sides to the power $1/p^n$, rearranging, and noting that $g_n(p)$ is increasing in n, we see that

$$t^{(p^n-1)/[(p-1)p^n]} P_t\phi(x) \leq u(t,x)^{1/p^n} D^{(p^n-1)/[(p-1)p^n]} \prod_{j=2}^{\infty} (1+p+\cdots+p^{j-1})^{1/p^j}.$$

But

$$\log \prod_{j=2}^{\infty} (1 + p + \cdots + p^{j-1})^{1/p^j} \leq \log \prod_{j=2}^{\infty} (jp^j)^{1/p^j} = \sum_{j=2}^{\infty} \frac{1}{p^j} \log (jp^j) < \infty$$

by the ratio test, since the ratio of consecutive terms is

$$\frac{p^j \log ((j+1)p^{j+1})}{p^{j+1} \log (jp^j)} = \frac{\log(j+1) + j \log p + \log p}{p(\log j + j \log p)} \xrightarrow[j \to \infty]{} \frac{1}{p} < 1,$$

and the theorem follows by letting $n \to \infty$ since $u(t,x)$ is essentially bounded. □

We now prove the main non-existence theorem, based on [12, Theorem 2.2].

Theorem 6.2 (Non-existence)**.** *Assume that k satisfies (K1)-(K5) and the left-hand inequality of (K6). As in Proposition 6.1, assume that there is a constant $D > 0$ such that for all $x \in G, U \geq 0$ we have $f(x,U) \geq DU^p$. Then if $1 < p \leq 1 + 2/d_s$ and $\phi(x) \geq 0$ for all $x \in G$ but ϕ is not identically zero, then* (6.2) *has no non-negative essentially bounded global solutions.*

Proof. Assume that we have such a solution $u(t,x)$ and that ϕ is not identically zero. The lower bound on the heat kernel and the monotone convergence theorem give

$$t^{d_s/2} \int_G k(t,x,y)\phi(y)d\mu(y) \geq a_1 \int_G \exp\left(-c_1 \left(\frac{||x-y||^{d_w}}{t}\right)^{\frac{1}{d_w - 1}}\right)\phi(y)d\mu(y)$$

$$\xrightarrow[t \to \infty]{} a_1 \int_G \phi(y)d\mu(y) = a_1 ||\phi||_1 \quad (6.6)$$

even if $||\phi||_1 = \infty$. Combining this with (6.3) we see that there is $M_3 > 0$ such that $t^{(d_s/2) - 1/(p-1)} > M_3$ for all t sufficiently large, which means that $p \geq 1 + 2/d_s$.

Now consider the case $p = 1 + 2/d_s$. (6.3) gives

$$M_1 \geq t^{d_s/2} \int_G k(t,x,y)\phi(y)d\mu(y) \xrightarrow[t \to \infty]{} a_1 ||\phi||_1$$

so $||\phi||_1 < \infty$. Now for any $t_0 > 0, v(t,x) := u(t + t_0, x)$ is a solution of (6.2) with initial data $\psi(x) := u(t_0, x)$, so by the above procedure we have that there is $M_5 > 0$ such that for all $t > 0$ we have

$$\int_G u(t,y)d\mu(y) \leq M_5. \quad (6.7)$$

We next claim that for each $t_0 > 0$ there is $\epsilon > 0$ and $b_1 > 0$ depending on t and ϕ such that for all $x \in G$ we have

$$u(t_0, x) \geq b_1 k(\epsilon, x, 0). \quad (6.8)$$

To show this, let $\alpha := 1/(d_w - 1)$ and $\beta = d_w/(d_w - 1)$. By the upper bound on k, we have that

$$k(\epsilon, x, 0) a_2^{-1} \epsilon^{d_s/2} \exp\left(c_2 \frac{||x||^\beta}{\epsilon^\alpha}\right) \leq 1$$

so using (6.2) and the lower bound on k,

$$\begin{aligned} u(t_0,x) &\geq \int_G k(t_0,x,y)\phi(y)d\mu(y) \\ &\geq a_1 t_0^{-d_s/2}\int_G \exp\left(-c_1\frac{||x-y||^\beta}{t_0^\alpha}\right)\phi(y)d\mu(y) \\ &\geq \frac{a_1}{a_2}\left(\frac{\epsilon}{t_0}\right)^{d_s/2} k(\epsilon,x,0)\int_G \exp\left(\frac{c_2}{\epsilon^\alpha}||x||^\beta - \frac{c_1}{t_0^\alpha}||x-y||^\beta\right)\phi(y)d\mu(y). \end{aligned} \tag{6.9}$$

Now, $d_w > 1$ so $\beta > 1$ so the function $t \mapsto t^\beta$ is convex on the non-negative reals. Therefore, for all $x, y \in \mathbb{R}^n$ we have

$$||x-y||^\beta \leq (||x|| + ||y||)^\beta = 2^\beta\left(\frac{1}{2}||x|| + \frac{1}{2}||y||\right)^\beta \leq 2^\beta\left(\frac{1}{2}||x||^\beta + \frac{1}{2}||y||^\beta\right) = 2^{\beta-1}(||x||^\beta + ||y||^\beta)$$

so if we choose ϵ such that $c_2\epsilon^{-\alpha} = 2^{\beta-1}c_1 t_0^{-\alpha}$ then

$$c_2||x||^\beta\epsilon^{-\alpha} - c_1||x-y||^\beta t_0^{-\alpha} \geq \left(c_2\epsilon^{-\alpha} - 2^{\beta-1}c_1 t_0^{-\alpha}\right)||x||^\beta - 2^{\beta-1}c_1||y||^\beta t_0^{-\alpha} = -2^{\beta-1}c_1||y||^\beta t_0^{-\alpha}.$$

Thus (6.8) follows from (6.9) upon taking

$$b_1 = \frac{a_1}{a_2}\left(\frac{\epsilon}{t_0}\right)^{d_s/2}\int_G \exp\left(-2^{\beta-1}c_1 t_0^{-\alpha}\right)\phi(y)d\mu(y).$$

To finish the proof with $p = 1 + 2/d_s$, define $v(t,x) := u(t+1,x)$. Clearly $v(t,x)$ satisfies (6.2) with initial data $u(1,x)$ instead of $\phi(x)$. By (6.2) and (6.8) and the semigroup property of the heat kernel, we have that

$$v(t,x) \geq \int_G k(t,x,y)u(1,y)d\mu(y) \geq b_1\int_G k(t,x,y)k(\epsilon,y,0)d\mu(y) = b_1 k(t+\epsilon,x,0)$$

so by (6.2) and the normalisation property of k, we have

$$\begin{aligned} \int_G v(t,x)d\mu(x) &\geq \int_G d\mu(x)\int_0^t\int_G k(t-\tau,x,y)f(y,v(\tau,y))d\mu(y) \\ &\geq \int_G d\mu(x)\int_0^t\int_G k(t-\tau,x,y)Dv(\tau,y)^p d\mu(y) \\ &\geq D\int_0^t d\tau\int_G v(\tau,y)^p d\mu(y) \\ &\geq Db_1^p\int_0^t d\tau\int_G k(\tau+\epsilon,y,0)^p d\mu(y). \end{aligned} \tag{6.10}$$

By the bounds on k and the particular value $p = 1 + 2/d_s$, we have

$$\begin{aligned} k(\tau+\epsilon,y,0)^p &\geq a_1^p(\tau+\epsilon)^{-pd_s/2}\exp(-pc_1||y||^\beta(\tau+\epsilon)^{-\alpha}) \\ &= a_1^p\left(\frac{c_2}{pc_1}\right)^{d_s/2\alpha}(\tau+\epsilon)^{-1}\left(\frac{\tau+\epsilon}{(pc_1c_2^{-1})^{1/\alpha}}\right)^{-d_s/2}\exp\left(-\frac{c_2||y||^\beta}{c_2(\tau+\epsilon)^\alpha/pc_1}\right), \\ &\geq \frac{a_1^p}{a_2}\left(\frac{c_2}{pc_1}\right)^{d_s/2\alpha}(\tau+\epsilon)^{-1}k((\tau+\epsilon)(pc_1c_2^{-1})^{-1/\alpha},y,0). \end{aligned}$$

This, together with (6.10), shows that there is $M_6 > 0$ such that

$$\int_G v(t,x)d\mu(x) \geq M_6 \int_0^t (\tau+\epsilon)^{-1}d\tau \xrightarrow[t\to\infty]{} \infty,$$

contradicting (6.7) and proving the theorem.

□

We remark that the case $p < 1 + 2/d_s$ of Theorem 6.2 follows from the more delicate case $p = 1 + 2/d_s$. This is because if $p < 1 + 2/d_s$ and $f(x,U) \geq DU^p$ then $U^p/U^{1+2/d_s} \to \infty$ as $U \to 0^+$ so $f(x,U) > EU^{1+2/d_s}$ for some $E > 0$ (as this is true on $(0,\epsilon)$ for some sufficiently small ϵ, and also true on (ϵ,T) because DU^p and U^{1+2/d_s} are bounded away from 0 and ∞ on (ϵ,T)). However, we have included the direct proof in the case $p < 1 + 2/d_s$ because it is simpler than the critical case $p = 1 + 2/d_s$. As already mentioned, there are heat kernels on the unbounded Sierpinski gasket and unbounded hexagasket that satisfy the properties that ensure that the above theorems, and the next theorem, will hold. It is important to emphasise how crucially these theorems depend on the spectral dimension of the domain, which is an analytic property of the self-similar structure. In some sense, p takes the place of the dimension n in the classical situation of the heat equation on $\mathbb{R}^n$ (see [48], which considers the case $f(x,u(x)) = u^p$ on $\mathbb{R}^n$).

6.2 Existence

We now prove a general existence result that we will show applies in our situation if the initial data are sufficiently small, adapting [48, Theorem 3] and [12, Theorem 3.1]. It states, roughly, that if the initial data ϕ is such that the solution of the corresponding *linear* equation decays sufficiently fast with time, then there is a solution of the *semilinear* equation with the same initial condition. Note that the following theorem holds for all $p > 1$, though we will only be able to apply it in Corollary 6.3.1 for certain values of p. Again note the uniformity of the constant D.

Theorem 6.3. *[Existence] Assume that k satisfies (K1)-(K5). Let $D > 0$, $1 < p < \infty$, $1 \leq q < \infty$. Suppose that for all $x \in G, U \geq 0$ we have $0 \leq f(x,U) \leq DU^p$, and that for each $x \in G$, $U \mapsto f(x,U)$ is continuous and increasing in U. Suppose further that $\phi \geq 0$ satisfies $\phi \in L^q(G)$ and*

$$\int_0^\infty ||P_\tau\phi||_\infty^{p-1}d\tau \leq \frac{1}{D(p-1)}. \tag{6.11}$$

Then (6.2) has a non-negative global solution u such that for all $T > 0$, $u(t,\cdot)$ is bounded in $L^q(G)$, uniformly for $t \in (0,T)$.

Proof. Define $b : [0,\infty) \to \mathbb{R}$ by

$$b(t)^{-(p-1)} = 1 - D(p-1)\int_0^t ||P_\tau\phi||_\infty^{p-1}d\tau.$$

Note that this is well-defined because by (6.11) the right-hand side always lies in $[0,1]$. Note that $b(0) = 1$ and by the chain rule $b'(t) = Db(t)^p||P_t\phi||_\infty^{p-1} > 0$. Therefore b is increasing, so $b(t) \geq 1$ for

all $t \geq 0$, and b satisfies the integral equation

$$b(t) = 1 + D \int_0^t b(\tau)^p ||P_\tau \phi||_\infty^{p-1} d\tau.$$

Now let $u : [0, \infty) \times G \to \mathbb{R}$ be any continuous function such that for all $t \geq 0$ and $x \in G$ we have

$$P_t\phi(x) \leq u(t, x) \leq b(t)P_t\phi(x).$$

Define the operator F by

$$Fu(t, x) = P_t\phi(x) + \int_0^t P_{t-\tau} f(x, u(\tau, x))d\tau. \tag{6.12}$$

Then for all $t \geq 0$ and $x \in G$, using the integral equation for b and properties of the heat kernel, we have

$$\begin{aligned} Fu(t, x) &\leq P_t\phi(x) + \int_0^t d\tau \int_G k(t - \tau, x, y)D[u(\tau, x)]^p d\mu(y) \\ &\leq P_t\phi(x) + D \int_0^t d\tau \int_G k(t - \tau, x, y)b(\tau)^p [P_\tau\phi(x)]^p d\mu(y) \\ &\leq P_t\phi(x) + D \int_0^t b(\tau)^p ||P_\tau\phi||_\infty^{p-1} d\tau \int_G k(t - \tau, x, y)P_\tau\phi(y)d\mu(y) \\ &= b(t)P_t\phi(x). \end{aligned}$$

Therefore for all $t \geq 0$ and $x \in G$ we have $P_t\phi(x) \leq Fu(t, x) \leq b(t)P_t\phi(x)$. Inductively define $u_0(t, x) = P_t\phi(x)$ and $u_{m+1}(t, x) = Fu_m(t, x)$ for $m = 0, 1, 2, \ldots$. Applying the above argument inductively we see that

$$P_t\phi(x) \leq u_m(t, x) \leq b(t)P_t\phi(x) \quad \text{for all} \quad m \geq 0. \tag{6.13}$$

For any $t > 0, x \in G$, clearly $u_0(t, x) \leq u_1(t, x)$, and by the definition of F and the monotonicity of f we have that if $v_1(t, x) \leq v_2(t, x)$ then $Fv_1(t, x) \leq Fv_2(t, x)$. So by induction, the sequence $u_m(t, x)$ is increasing in m. Therefore there exists a measurable function $u(t, x)$ such that for all $t > 0, x \in G$ we have

$$u_m(t, x) \xrightarrow[m\to\infty]{} u(t, x) \in [0, \infty] \qquad \text{with} \qquad P_t\phi(x) \leq u(t, x) \leq b(t)P_t\phi(x)$$

Since f is continuous in the second variable, $f(y, u_m(\tau, y)) \to f(y, u(\tau, y))$ for all $\tau \geq 0, y \in G$. Therefore by the monotone convergence theorem, for all $t \geq 0, x \in G$ we have

$$\int_0^t d\tau \int_G k(t - \tau, x, y)f(y, u_m(\tau, y))d\mu(y) \xrightarrow[m\to\infty]{} \int_0^t d\tau \int_G k(t - \tau, x, y)f(y, u(\tau, y))d\mu(y).$$

Taking the limit $m \to \infty$ in

$$u_{m+1}(t, x) = Fu_m(t, x) = P_t\phi(x) + \int_0^t d\tau \int_G k(t - \tau, x, y)f(y, u_m(\tau, y))d\mu(y)$$

we see that $u = Fu$, i.e. u satisfies (6.2). By (6.13) and the contraction property of the heat kernel, we see that

$$||u(t,\cdot)||_q \leq b(t)||P_t\phi||_q \leq b(t)||\phi||_q$$

so if $t > 0$, since $b(t)$ is continuous hence bounded on $[0,T]$, $u(t,\cdot)$ is bounded in $L^q(G)$, uniformly for $t \in (0,T)$.

□

Note, as noted in [48, Remark (5)], that if the hypothesis (6.11) is weakened to the assertion that there exists an $\epsilon > 0$ such that

$$\int_0^\epsilon ||P_\tau\phi||_\infty^{p-1} d\tau \leq \frac{1}{D(p-1)}$$

then by the above proof, there is a solution of (6.2) for all time t such that

$$\int_0^t ||P_\tau\phi||_\infty^{p-1} d\tau \leq \frac{1}{D(p-1)}.$$

The following corollary gives the required existence result in our situation, based on [12, Corollary 3.2], fixing a minor error ('0' was used in place of z, which does not make sense as '0' is not an element of G). Here, the condition $p > 1 + 2/d_s$ is crucial, unlike in the previous theorem which holds for all $p > 1$.

Corollary 6.3.1. *Assume that k satisfies (K1)-(K6). Let $p > 1 + 2/d$ and suppose that f satisfies the same conditions as in the statement of Theorem 6.3. Let $\gamma > 0$ and suppose that $\phi \geq 0$. Then there exists $\delta > 0$ such that the following property holds: if $z \in G$ is such that*

$$0 \leq \phi(x) \leq \delta k(\gamma, x, z) \qquad \text{for } \mu\text{-almost all } x \in G$$

then (6.2) *has a non-negative global solution with the same boundedness property as in Theorem 6.3.*

Proof. By the upper bound on k and the semigroup property,

$$||P_t\phi||_\infty = \sup_{x\in G} \int_G k(t,x,y)\phi(y)d\mu(y) \leq \delta \sup_{x\in G} k(t+\gamma,x,z) \leq a_2\delta(t+\gamma)^{-d_s/2}.$$

Therefore if δ is sufficiently small then

$$\int_0^\infty ||P_t\phi||_\infty^{p-1} dt \leq (a_2\delta)^{p-1} \int_0^\infty (t+\gamma)^{-(p-1)d_s/2} dt \leq \frac{1}{D(p-1)}$$

since the integral is finite as $(p-1)d_s/2 > 1$ since $p > 1 + 2/d_s$. Therefore there is a global solution with the required properties by Theorem 6.3. □

Next, we give a more precise version of the remark at the end of the statement of [12, Corollary 3.2].

Corollary 6.3.2. *Assume that k satisfies (K1)-(K6). Let $p > 1 + 2/d_s$ and suppose that f satisfies the same conditions as in the statement of Theorem 6.3. Suppose $\psi : G \to \mathbb{R}$ is measurable and non-negative with compact support $K \subseteq G$. Suppose that ψ is essentially bounded, with essential supremum S, say. Then there is a constant C such that if $c \leq C$ then* (6.2) *has a non-negative global solution with initial data $\phi := c\psi$, with the same boundedness property as in as in Theorem 6.3.*

Proof. Let $\gamma > 0$, let $\delta > 0$ be as in Corollary 6.3.1, and let $z \in G$. As K is compact and k is continuous and $k(\gamma, x, z) > 0$ for all $x, z \in G$, $x \mapsto k(\gamma, x, z)$ attains its infemum I on K, and $I > 0$. Let $C = \delta I/S$, let $c \leq C$, and let $\phi = c\psi$. Then

$$0 \leq \phi(x) \leq C\psi(x) \leq \delta I \leq \delta k(\gamma, x, z)$$

for μ-almost every $x \in G$, so by Corollary 6.3.1 there exists a solution with the required properties. □

Falconer and Hu [12] give an alternative condition that can be placed on ϕ to allow Theorem 6.3 to be applied in our situation.

Corollary 6.3.3. *Assume that k satisfies (K1)-(K6). Let $d_s \leq 2$ and $p > 1 + 2/d_s$ and suppose that f satisfies the same conditions as in the statement of Theorem 6.3. If $||\phi||_{d_s(p-1)/2}$ is sufficiently small then we can use a result known as the Marcinkiewicz Interpolation Theorem that bounds the norms of nonlinear operators acting on L^p spaces to show that the condition (6.11) is satisfied, and hence by Theorem 6.3 there exists a non-negative global solution of (6.2) with the required boundedness properties.*

Proof. See [12, pp. 615-7]. □

6.3 Regularity

Let $T > 0$ and assume that a bounded solution $u(t, x)$ of (6.2) exists on $(0, T) \times G$ for some ϕ and f, say $|u(t, x)| < M$ for $t \in (0, T), x \in G$. We now investigate the regularity of such solutions, first considering Hölder continuity in x using the relevant result for the linear equation, Theorem 5.5. The following theorem and proof are based on the second part of the proof of [12, Theorem 4.3], but here we include the case $\nu = 1$.

Theorem 6.4. *[Hölder continuity] Suppose that the measure μ satisfies the condition*

$$\exists m > 0 \quad \forall r > 0 \, \forall x \in G \quad \mu(B_r(x)) \leq m r^{d_f} \qquad (*)$$

where d_f is the same constant as in (K6) (recall that this holds if G is self-similar with Hausdorff dimension d_f and μ is the restriction of d_f-dimensional Hausdorff measure to G). Assume that k satisfies (K1)-(K7). Assume that $\phi \in L^1(G) \cap L^\infty(G)$ is Hölder continuous with exponent $\lambda \in (0, 1]$, i.e. there exists $C_0 > 0$ such that for all $x_1, x_2 \in G$ we have

$$|\phi(x_2) - \phi(x_1)| \leq C_0 ||x_2 - x_1||^\lambda.$$

In addition, assume that $T > 0$ is such that there is $C_f > 0$ such that if $x \in G$ and $t \in (0, T)$ then

$$|f(x, u(t, x))| \leq C_f. \tag{6.14}$$

Then there is $C_2 > 0$ (depending on T) such that for any solution u of (6.2) that is bounded on $(0, T) \times G$ with $||u(t, \cdot)||_1$ bounded on $(0, T)$, we have that for all $t \in (0, T)$ and $x_1, x_2 \in G$,

$$|u(t, x_2) - u(t, x_1)| \leq C_2 ||x_2 - x_1||^\gamma$$

holds, where $\gamma := \lambda\sigma/(\lambda + \nu d_w)$, where d_w is as in (K6) (usually the walk dimension) and σ is also as in (K6). This is a Hölder condition in x, uniformly for $t \in (0, T)$.

Proof. Define

$$w(t,x) = \int_0^t d\tau \int_G k(t-\tau, x, y) f(x, u(\tau, y)) d\mu(y).$$

Let η and t be such that $0 < \eta < t/2 < t < T$. Then if $x \in G$, then by the renormalisation property,

$$\int_{t-\eta}^t d\tau \int_G k(t-\tau, x, y) f(y, u(\tau, y)) d\mu(y) \leq \int_{t-\eta}^t d\tau \int_G k(t-\tau, x, y) C_f d\mu(y) \leq \int_{t-\eta}^t C_f d\tau \leq C_f \eta.$$

First consider the case $\nu > 1$. If $x_1, x_2 \in G$ are such that $||x_2 - x_1|| \leq 1$ then, using (K7),

$$\begin{aligned}
|w(t,x_2) - w(t,x_1)| =& \Big| \int_{t-\eta}^t d\tau \int_G k(t-\tau, x_2, y) f(y, u(\tau, y)) d\mu(y) \\
&- \int_{t-\eta}^t d\tau \int_G k(t-\tau, x_1, y) f(y, u(\tau, y)) d\mu(y) \\
&+ \int_0^{t-\eta} d\tau \int_G [k(t-\tau, x_2, y) - k(t-\tau, x_1, y)] f(y, u(\tau, y)) \Big| \\
\leq& 2C_f\eta + \left| M_0 \int_0^{t-\eta} d\tau \int_G |t-\tau|^{-\nu} ||x_2 - x_1||^\sigma f(y, u(\tau, y)) \right| \\
\leq& 2C_f\eta + M_0(t-\eta) C_f (T^{1-\nu} + \eta^{1-\nu}) ||x_2 - x_1||^\sigma \\
\leq& 2C_f\eta + 2M_0 T C_f \eta^{1-\nu} ||x_2 - x_1||^\sigma \\
\leq& C(\eta + \eta^{1-\nu} ||x_2 - x_1||^\sigma)
\end{aligned}$$

where $C = \max(C_f, 2M_0 T C_f)$. Now, $|w(t,x_2) - w(t,x_1)|$ is independent of η, so the above chain of inequalities holds in particular when η is sufficiently small so that $0 < \eta \leq \min(t/2, ||x_2 - x_1||^{\sigma/\nu})$ to give

$$|w(t,x_2) - w(t,x_1)| \leq C(||x_2 - x_1||^{\sigma/\nu} + ||x_2 - x_1||^{\sigma + (1-\nu)\sigma/\nu}) \leq 2C||x_2 - x_1||^{\sigma/\nu}.$$

Now, $\lambda \leq 1 < d_w$ so $\sigma/\nu \geq \gamma$, so w also satisfies a Hölder condition with exponent γ. Putting this estimate, together with the estimate obtained in Theorem 5.5 into (6.2) gives the result (for $\nu > 1$) when $||x_2 - x_1|| \leq 1$, and this extends to all $x_1, x_2 \in G$ since u is bounded.

Now consider the case $\nu = 1$. Note that $\lambda \leq 1 < d_w$ so $\sigma = \sigma/\nu > \gamma$, so $|z^\sigma \log z| = o(z^\gamma)$ as $z \to 0^+$. Therefore if $x_1, x_2 \in G$ are such that $||x_2 - x_1|| \leq 1$ then, again using (K7) and letting $0 < \eta \leq \min(t/2, ||x_2 - x_1||^{\sigma/\nu})$, there are constants C and C' such that

$$\begin{aligned}
|w(t,x_2) - w(t,x_1)| &\leq 2C_f\eta + \left| M_0 \int_0^{t-\eta} d\tau \int_G |t-\tau|^{-1} ||x_2 - x_1||^\sigma f(y, u(\tau, y)) \right| \\
&\leq C(\eta + |\log(\eta)| ||x_2 - x_1||^\sigma) \\
&\leq C(||x_2 - x_1||^\sigma + ||x_2 - x_1||^\sigma \log(||x_2 - x_1||^\sigma)) \\
&= C(||x_2 - x_1||^\sigma + \sigma ||x_2 - x_1||^\sigma \log ||x_2 - x_1||) \\
&\leq C' ||x_2 - x_1||^\gamma
\end{aligned}$$

where the constant C may now be different to in the case $\nu > 1$. The result follows as in the previous case.

□

Note that, since u is assumed to be bounded, there are many different possible conditions on f that would ensure that (6.14) holds, for example any of the following:

(1) f is bounded.

(2) f is independent of x and continuous (then by the extreme value theorem (6.14) holds).

(3) f satisfies conditions (6.19) and (6.17) from Theorem (6.7) (see the start of the proof of Theorem (6.7) to see why (6.14) then holds).

For example, $f(x, U) = |U|^p$ satisfies conditions (2) and (3).

Theorem 6.4 relates to regularity (Hölder continuity) of $u(t, x)$ in the x variable. We now investigate regularity (Lipschitz continuity and μ-almost everywhere differentiability) of $u(t, x)$ in t. This requires two general theorems in analysis:

Proposition 6.5 (Special case of Grönwall's inequality)**.** *Let $d, T > 0$ and $c \geq 0$. Suppose $g : [0, T] \to \mathbb{R}$ is measurable and bounded and satisfies $g(t) \leq c + d \int_0^t g(\tau) d\tau$ whenever $0 \leq t < T$. Then*

$$g(t) \leq c \exp(dt)$$

Proof. The proof of a more general version is in [9, p. 98]. □

Proposition 6.6 (Case of Rademacher's theorem)**.** *Let $U \subseteq \mathbb{R}$ be open and let $g : U \to \mathbb{R}$ be Lipschitz. Then g is differentiable at Lebesgue-almost every point in U, with derivative bounded by the Lipschitz constant.*

Proof. The proof of a more general version is in [21, pp. 46-48]. □

We now show that, under some assumptions, if the initial data ϕ is such that the solution of the *linear* equation satisfies a Lipschitz condition, then the solution of the *semilinear* equation will too. We follow the structure of the proof given for nonlinear term u^p in [12, Theorem 4.4], but with a much more general nonlinear term and elaborating on parts of the proof:

Theorem 6.7. *[Lipschitz continuity] Assume that k satisfies (K1)-(K5). Assume that the initial data ϕ satisfies $||\phi||_1 < \infty$ and that $P_t\phi$ is Lipschitz in t, uniformly for $x \in G$, i.e. there is $c_0 \geq 0$ such that for all $t > 0, \delta > 0, x \in G$ we have*

$$|P_{t+\delta}\phi(x) - P_t\phi(x)| \leq c_0\delta. \tag{6.15}$$

Assume that $f(x, U)$ is locally Lipschitz in U, uniformly for x, i.e. for all $U \in \mathbb{R}$, there are $\epsilon_U > 0$ and $C_U > 0$ such that if $x \in G$ and $U_1, U_2 \in \mathbb{R}$ are such that $|U_1 - U| < \epsilon_U$ and $|U_2 - U| < \epsilon_U$, then

$$|f(x, U_2) - f(x, U_1)| \leq c_U|U_2 - U_1|. \tag{6.16}$$

Assume further that there is $C_{zero} > 0$ such that for all $x \in G$ we have

$$f(x, 0) \leq C_{zero}. \tag{6.17}$$

Assume that $u(t, x)$ is a bounded solution, i.e. there is $M > 0$ such that for all $t > 0, x \in G$ we have

$$|u(t, x)| \leq M \tag{6.18}$$

Then for all $T > 0$, $u(t, x)$ is Lipschitz in t, uniformly for x, on $(0, T) \times G$.

Proof. First note that by compactness of $[-(M+1), M+1]$ there are finitely many points $U_i \in [-(M+1), M+1]$ such that the neighbourhoods $(U_i - \epsilon_{U_i}, U_i + \epsilon_{U_i})$ cover $[-(M+1), M+1]$. Let $C_{max} = \max C_{U_i}$. Then by repeated application of the triangle inequality and (6.16), $f(x, U)$ is Lipschitz in U for $U \in [-(M+1), M+1]$ with Lipschitz constant C_{max}, uniformly for $x \in G$, so if $|U| \leq M$ and $|\delta| \leq 1$ then for all $x \in G$ we have

$$|f(x, U + \delta) - f(x, U)| \leq C_{max}|\delta|. \tag{6.19}$$

Therefore by (6.17) and (6.18), for all $t > 0, x \in G$ we have

$$|f(x, u(t, x))| \leq C_{zero} + MC_{max} =: C_f. \tag{6.20}$$

By the substitution $\tau \mapsto t - \tau$, (6.2) is equivalent to

$$u(t, x) = \int_G k(t, x, y)\phi(y)d\mu(y) + \int_0^t d\tau \int_G k(\tau, x, y)f(x, u(t - \tau, x))d\mu(y). \tag{6.21}$$

Therefore by (6.15), (6.19), (6.18), (6.20) and the renormalisation property of k, we have that for all $t, \delta > 0, x \in G$,

$$\begin{aligned}
|u(t + \delta, x) - u(t, x)| \leq & |P_{t+\delta}\phi(x) - P_t\phi(x)| \\
& + \int_t^{t+\delta} d\tau \int_G k(\tau, x, y)|f(y, u(\delta + t - \tau, y))|d\mu(y) \\
& + \int_0^t d\tau \int_G k(\tau, x, y)|f(y, u(\delta + t - \tau, y)) - f(y, u(t - \tau, y))|d\mu(y) \\
\leq & c_0\delta + C_f\delta + C_{max} \int_0^t d\tau \int_G k(\tau, x, y)|u(\delta + t - \tau, y) - u(t - \tau, y)|d\mu(y).
\end{aligned}$$

So letting $g(t) := \sup_{x \in G} |u(t + \delta, x) - u(t, x)|$ for $t > 0$, we have that

$$g(t) \leq (c_0 + C_f)\delta + C_{max} \int_0^t g(t - \tau)d\tau \leq (c_0 + C_f)\delta \exp(C_{max}t)$$

by the renormalisation property of k and Grönwall's inequality. So for $T > 0$, letting $b = (c_0 + C_f) \exp(C_{max}T)$, we have

$$|u(t + \delta, x) - u(t, x)| \leq b\delta$$

for all $t \in (0, T), x \in G$, which is the required Lipschitz condition. □

Note that the bound (6.20) is all that we use (6.17) for, so it is clear that (6.17) could be replaced by a number of other conditions (for example replacing '0' with any $U \in \mathbb{R}$ in (6.17)) that would ensure that (6.20) holds.

Corollary 6.7.1. *Under the same conditions as in Theorem 6.7, for all $x \in G$, $\frac{\partial u}{\partial t}(t,x)$ exists for Lebesgue-almost every $t \in (0,T)$, and is bounded by a constant independent of x and $t \in (0,T)$.*

Proof. Fix $x \in G$ and apply Rademacher's theorem to the function $t \mapsto u(t,x)$, noting the uniformity of the Lipschitz constant in Theorem 6.7. □

An example of a condition on k and a form for ϕ that will ensure that the condition (6.15) holds is [12, p. 622], as (6.15) may be difficult to verify directly.

7 Conclusion

We have seen that there are several different approaches to defining a Laplacian on different fractal domains, most of which were developed by adapting known approaches on classical domains. Once a Laplacian has been defined, this gives rise to a rich theory of harmonic structure, spectral theory, and non-existence, existence and regularity of PDEs on fractal domains. There are many similarities with the theory that was long ago developed for classical domains, but some differences too, and many results are considerably harder to prove on fractal domains. Many results, including heat kernel bounds and consequently many of the results on the semilinear equation that we have obtained in Chapter 6, depend crucially on geometrical and analytical properties of the self-similar structure on which the process is occurring, encapsulated by constants such as the Hausdorff, spectral and walk dimensions.

As noted in the introduction, there is much work to do to determine how closely the mathematical theories explored here model physical processes on real-world fractal-like objects, but this is more a physics question than a mathematical one. There are two natural ways that the mathematical theory of PDEs on fractals could be extended further: either by considering different/more general PDEs (or trying to model different/more general physical processes), or by working on different/more general domains.

Even on classical domains, there are many partial differential equations for which many questions remain unanswered. Different PDEs are often analysed in very different ways. Since the situation is obviously harder on fractal domains, and considering the plethora of physical processes that can be modelled by PDEs (see Chapter 3.1), much more work needs to be done to analyse different PDEs on fractal domains. One example of a PDE involving a Laplacian that has not been studied in nearly as much depth as the heat equation is the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$ for constant c. Barlow [4] notes that the "wave equation is rather harder, since it is not very susceptible to probabilistic analysis" but that it has been analysed in [28] on some globally fractal-like manifolds.

With regard to considering PDEs on more general domains, analysis is often significantly harder on sets that are not finitely ramified, and the methods used will often need to be quite different.

Only certain classes of fractals have yet been studied; Bass [6] discusses some of the analysis that has been done on the Sierpinski carpet. Recently, there has been much interest in the analysis of PDEs on metric measure spaces (defined in (2.11)). In [13] for example, the semilinear equation $\partial u/\partial t = \Delta u + u^p + f(x)$ for appropriate functions f is considered on arbitrary metric measure spaces. In such a general setting, the methods of defining a Laplacian pointwise by approximating graphs will not be possible; in [13], as in most of Section 6, the authors assume the existence of a heat kernel and use it to define the meaning of a weak solution of the above PDE. Again, this often requires Gaussian-type bounds on the heat kernel; these bounds have been studied in their own right in much detail by, for example, Grigor'yan and Telcs in [20].

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